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CONSERVATION LAWS AND ABSOLUTE PARALLELISM IN GENERAL RELATIVITY

BY

C. MØLLER

København 1961
i kommission hos Ejnar Munksgaard

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Synopsis

It is shown that a satisfactory solution of the problem of the energy distribution in a gravitational field can be obtained in the framework of a tetrad formulation of the gravitational theory. Besides the usual gravitational field equations, a set of six supplementary equations, needed for a unique determination of the energy-momentum complex, is formulated and discussed in detail. The results is a revival of Einstein's old idea of „Fernparallelismus" and the discovery of new geometrical properties of the space-time continuum.

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1. Introduction and Summary

Shortly after EINSTEIN had put forward his general theory of relativity in 1916 the question of the energy of the gravitational field was discussed in a number of papers by various authors (1). EINSTEIN himself was able to define an "energy-momentum complex" Θ_i^k, depending algebraically on the metric tensor and its derivatives, which satisfies the local conservation law and gives a satisfactory description of the total energy and momentum of a closed system. In this connection it is essential that the expressions for the total momentum and energy, viz.

$$P_i = \frac{1}{c} \underset{x^4 = \text{const.}}{\iiint} \Theta_i^4 \, dx^1 \, dx^2 \, dx^3, \tag{1.1}$$

are constant in time and are transformed like the components of a free 4-vector under linear space-time transformations. On the other hand, the integral

$$-\underset{V}{\iiint} \Theta_4^4 \, dx^1 dx^2 dx^3 \tag{1.2}$$

extended over a finite volume V in 3-space depends particularly on the choice of spatial coordinates (x^1, x^2, x^3) used in the evaluation of the integral, i. e. the quantity (1.2) cannot be interpreted unambiguously as the energy content of the volume V. The conclusion of these early investigations was, therefore, that it had no well-defined physical meaning to speak of a definite distribution of the gravitational energy throughout space. As a consequence, *no* gravitational analogue to Pointing's theorem, which regulates the distribution and flow of energy in an electromagnetic field, could be formulated by means of the complex Θ_i^k. This circumstance made it difficult to treat problems of gravitational radiation in an unambiguous way. Moreover, although we have to be prepared for the possibility that familiar notions of ordinary field theory may lose their meaning in general relativity, it seems rather strange that a simple notion like the energy content of our laboratory should have no exact meaning because the laboratory is placed in the gravitational field of the earth.

For these reasons the discussion was taken up again a few years ago. In a paper in the Annals of Physics (2) from 1958 it was shown that another complex $\mathcal{T}_i^k$ could be defined which seemed to satisfy all requirements, in particular the conditions of localizability of the energy and of local conservation. Moreover, in a series of papers by different authors(3) this particular complex $\mathcal{T}_i^k$ was derived also by application of the method of infinitesimal transformations (Noether's theorem) to the curvature scalar density, which is the variant in the variational principle of the gravitational field equations. Thus it seemed that a final solution of the energy problem had been reached and that the gravitational field possessed more of the usual properties of fields than had hitherto been assumed.

However, in a recent paper(4) it was shown that the complex $\mathcal{T}_i^k$ suffers from a serious deficiency which makes it impossible in general to use $\mathcal{T}_i^k$ as the energy-momentum complex. In fact it was found that the quantities obtained from (1.1) by substitution of $\mathcal{T}_i^k$ for Θ_i^k do not have the right transformation properties under all linear transformations. Besides, it could be shown that no complex at all exists which is an algebraic function of the metric tensor and its derivatives and satisfies all the aforementioned requirements, including that of localizability of the energy.

On the other hand, it was indicated that a way out of this dilemma might possibly be found if the usual description of gravitational fields by means of the metric tensor was replaced by a "tetrad formulation". In the present paper it is shown that this idea may be developed into a consistent scheme that furnishes a solution to the problem of the energy distribution in gravitational fields and at the same time throws new light on earlier attempts at a unification of gravitation and electromagnetism.

In the next section, the energy-momentum complex T_i^k is defined as a function of the tetrad field functions $\underset{a}{h}^i(x)$ and their space-time derivatives of the first and second orders. It is proved that, for a suitable choice of the tetrads, T_i^k satisfies all desirable requirements, including the localizability condition, and the condition that

$$P_i = \frac{1}{c} \underset{x^4 = \text{const.}}{\iiint} \mathsf{T}_i^4 \, dx^1 \, dx^2 \, dx^3 \tag{1.3}$$

is transformed like a 4-vector under arbitrary linear-coordinate transformations. While the metric is defined by ten functions $g_{ik}(x)$, the tetrad field contains sixteen functions $\underset{a}{h}^i(x)$. The latter functions determine the metric tensor g_{ik} uniquely, but for given $g_{ik}(x)$ there is a certain latitude in the choice of the tetrads $\underset{a}{h}^i$ corresponding to arbitrary independent rotations of the four unit vectors constituting the tetrads in the different space-time points. Now, it turns out that the complex $\mathsf{T}_i^k\left[\underset{a}{h}^i(x)\right]$ as a function of the tetrad functions $\underset{a}{h}^i(x)$ and their derivatives is not invariant under arbitrary independent rotations of the tetrad vectors. Therefore, in order to get a complex $\mathsf{T}_i^k(x)$ which is a unique function of the coordinates for a given physical system, we have to specify the relative orientation, in different points, of the tetrads entering the expression $\mathsf{T}_i^k\left[\underset{a}{h}^i(x)\right]$. Thus, besides the gravitational field equations of Einstein, which determine the metric, we obviously need a set of six independent supplementary equations in order to single out the preferred tetrads to be applied in the expression $\mathsf{T}_i^{\ k}\left[\underset{a}{h^i}(x)\right]$ for the energy-momentum complex.

Section 3 is devoted to the setting up of these supplementary equations. They turn out to be of the form

$$\varphi_{ik} = 0\,, \tag{1.4}$$

where φ_{ik} is an antisymmetric tensor which is an algebraic expression in the tetrads $\underset{a}{h^i}(x)$ and their derivatives of the first and second orders. Since the equations (1.4) are differential equations, it is also necessary to specify a set of boundary conditions at spatial infinity in order to obtain a unique fixation of the relative orientation of the preferred tetrads. For a rather wide class of physical systems, viz. insular systems, these boundary conditions are formulated in section 3.

In section 4, the case of weak gravitational fields is treated in detail. It is shown that the finally adopted supplementary equations together with the boundary conditions determine the relative orientation of the preferred tetrads uniquely. In "harmonic" coordinates the solutions of the gravitational field equations and the supplementary equations are particularly simple, and the same holds for the resulting energy-momentum complex $\mathsf{T}_i^{\ k}(x)$, but the theory is generally covariant, and there is no question of any preferred systems of space-time coordinates.

In section 5, the properties of the space-time continuum of the theory developed in sections 2–4 are briefly discussed. The decisive step in that development was the introduction of a definite tetrad lattice (or rather of a certain restricted class of tetrad lattices) consisting of tetrads in every point with a fixed relative orientation. This circumstance allows us to speak in an unambiguous way of parallelism of vectors at distant points when they have equal components with respect to the local tetrads at these points. Therefore, strictly speaking, the space-time continuum in this theory is not an ordinary Riemannian space, where the notion of parallelism at a distance has a meaning only with respect to a definite curve connecting the distant points. Space-time is here rather a space of the type considered first by WEITZENBÖCK[(5)], although it may always be pictured as a Riemannian space with a built-in tetrad lattice.

This circumstance puts one in mind of EINSTEIN's old idea of "Fernparallelismus", by which he hoped to arrive at a unified theory of gravitation and electromagnetism[(6)], and it suggests a rather natural generalization of the theory briefly discussed in the last section of this paper.

Since the number of tetrad functions $\underset{a}{h^i}(x)$ exceeds the number of the metric components $g_{ik}(x)$, it is quite natural to assume that the tetrad field actually describes a larger domain of physical phenomena than mere gravitation, and, on account of the excess number of variables being just six, one might assume that the sixteen variables $\underset{a}{h^i}(x)$ describe the unified field of gravitation and electromagnetism. Instead of regarding the equations (1.4) as supplementary field equations, as is done in sections 2–5, one should then rather think of them as expressing a particular physical situation, viz. the case where no electromagnetic fields are present. In general, the equation (1.4) would then have to be replaced by other equations, as for instance

$$\varphi_{ik} = F_{ik}(x), \tag{1.5}$$

where $F_{ik}(x)$ is the electromagnetic field tensor determined by Maxwell's equations in their usual general-relativity form.

It should be noted that the field equations arrived at in this way are entirely different from those of the unified theory of EINSTEIN. However, they are quite similar to a set of equations put forward at that time by LEVI-CIVITA(7). It still remains to be seen whether a generalization of the formalism developed in sections 2–5 along the lines indicated in section 6 can be carried through in a consistent way.

2. Survey of Earlier Results

In the paper(4) mentioned in the introduction it was shown that no "energy-momentum complex" $\mathsf{T}_i^{\ k}$ with the following properties exists:

1. $\mathsf{T}_i^{\ k}(x)$ is an affine tensor density depending algebraically on the components of the metric tensor g^{ik} and their derivatives of the first and second orders.
2. $\mathsf{T}_i^{\ k}$ satisfies the local conservation law

$$\mathsf{T}_i^{\ k},{}_k \equiv \frac{\partial \mathsf{T}_i^{\ k}}{\partial x^k} = 0 \tag{2.1}$$

identically.
3. For a closed system, where space-time is flat at spatial infinity and where we can use asymptotically rectilinear coordinates $(x^i) = (x^\iota, x^4)$, the quantities

$$P_i = \frac{1}{c} \iiint\limits_{x^4 = \text{const.}} \mathsf{T}_i^{\ 4}\, dx^1\, dx^2\, dx^3 \tag{2.2}$$

are constant in time, and they are transformed as the covariant components of a free vector under linear space-time transformations. This property is essential for the interpretation of $P_i = \{P_\iota - H/c\}$ as the total momentum and energy vector.
4. $\mathsf{T}^k = \mathsf{T}_4^{\ k}$ is transformed like a 4-vector density under the group of purely spatial transformations

$$\overline{x}^\iota = f^\iota(x^\varkappa), \quad \overline{x}^4 = x^4. \tag{2.3}$$

The last-mentioned property is necessary in order to make the energy content of a finite volume of space V, i. e.

$$H_V = -\iiint\limits_V \mathsf{T}_4^{\ 4}\, dx^1\, dx^2\, dx^3, \tag{2.4}$$

independent of the spatial coordinates used in the evaluation of the integral. Thus, 4. is the condition of localizability of the energy in a gravitational field.

The energy-momentum complex Θ_i^k given by Einstein is of the form

$$\Theta_i^k = \sqrt{-g}\,(T_i^k + \vartheta_i^k) = h_i^{kl}{}_{,l} \tag{2.5}$$

with the "superpotential"

$$h_i^{kl} = -h_i^{lk} = \frac{g_{in}}{2\varkappa\sqrt{-g}}\left[(-g)\,(g^{kn}g^{lm} - g^{ln}g^{km})\right]_{,m}. \tag{2.6}$$

Here T_i^k is the matter tensor which appears as the source of the gravitational field in Einstein's field equations

$$G_i^k \equiv R_i^k - \frac{1}{2}\delta_i^k R = -\varkappa T_i^k, \tag{2.7}$$

and ϑ_i^k is a homogeneous quadratic function of the first-order derivatives of the metric tensor. Thus, for a closed system, where the metric is given asymptotically by the Schwarzschild solution, we have for large spatial distances r

$$\vartheta_i^k \sim O\left(\frac{1}{r^4}\right), \quad \Theta_i^k \sim O\left(\frac{1}{r^4}\right). \tag{2.8}$$

Einstein's energy-momentum complex Θ_i^k has the properties 1.–3. Actually 3. is a consequence of 1. and 2. together with the asymptotic behaviour (2.8). However, Θ_i^k does *not* satisfy the localizability condition 4. For this reason Einstein came to the conclusion that, in general relativity, the energy content H_V of a finite part of space has no exact physical meaning.

On the other hand, the complex

$$\left.\begin{aligned} \mathcal{T}_i^k &= \chi_i^{kl}{}_{,l} \\ \chi_i^{kl} &= -\chi^{lk} = \frac{\sqrt{-g}}{\varkappa}\,(g_{in,m} - g_{im,n})\,g^{km}g^{ln} \end{aligned}\right\} \tag{2.9}$$

proposed in a previous paper[(2)] satisfies the conditions 1., 2. and 4., but *not* 3. As was shown in detail in [(4)], this is due to the fact that $\mathcal{T}_i^k$ does not show the asymptotic behaviour (2.8) for a closed system, but instead

$$\mathcal{T}_i^k \sim O\left(\frac{1}{r^3}\right) \tag{2.10}$$

for $r \to \infty$. Furthermore it was shown by M. Magnusson[(8)] that the complex (2.9) is the only expression satisfying 1., 2. and the localizability condition 4., and, as was shown in [(4)], Einstein's expression (2.5 – 6) is the only complex satisfying 1.–3. Therefore it is necessary to give up at least one of the properties 1.–4. Now, for physical reasons it seems hard to abandon the conditions 2.–4., so the only possibility left is to discard the condition 1. For instance, we might substitute for 1. the less specific assumption that

1′. $T_i^k(x)$ is an affine tensor density depending in a covariant way on a set of field variables that determine the gravitational field uniquely.

In the last section of (4) it was indicated how this might be done. The starting point of the considerations was the remark that the components g_{ik} of the metric tensor are probably not the truly fundamental field variables, since for instance the field equations for a fermion field in the presence of a gravitational field cannot be expressed in terms of these variables only. On the other hand, it is well known that in such cases the gravitational field may be described by "tetrads", i. e. by the components of four orthogonal unit vectors attached to every point in space-time. If we label these vectors with an index a running from 1 to 4, the contravariant components of the tetrad vectors are sixteen functions $\underset{a}{h}^i(x)$ of the coordinates (x^i), which may be regarded as the fundamental gravitational variables.

The covariant components of the tetrad vectors are

$$\underset{a}{h}_i = g_{ik}\,\underset{a}{h}^k. \tag{2.11}$$

One of the vectors, say $\underset{4}{h}^i$, is a time-like vector, the others $\underset{a}{h}^i$ are space-like; the orthonormality of the tetrads may then be written

$$\underset{a}{h}_i\,\underset{b}{h}^i = \eta_{ab}, \tag{2.12}$$

where η_{ab} is the constant 4×4 matrix

$$\left.\begin{aligned} \eta_{ab} &= \varepsilon_{a)}\,\delta_{ab} = \eta^{ab} \\ \varepsilon_a &= \{1,\,1,\,1,-1\}. \end{aligned}\right\} \tag{2.13}$$

(The parenthesis after the index a in (2.13) indicates that *no summation over a* should be performed!).

Along with the vectors $\underset{a}{h}^i$ we introduce the vectors

$$\overset{a}{h}{}^i = \eta^{ab}\,\underset{b}{h}^i = \varepsilon_{a)}\,\underset{a}{h}^i, \qquad \overset{a}{h}_i = \varepsilon_{a)}\,\underset{a}{h}_i. \tag{2.14}$$

Then (2.12) may also be written

$$\underset{a}{h}_i\,\overset{b}{h}{}^i = \delta_a^b \tag{2.15}$$

or

$$\overset{a}{h}_i\,\overset{b}{h}{}^i = \eta^{ab}. \tag{2.16}$$

Let

$$h = \det\left\{\underset{a}{h}_i\right\} \tag{2.17}$$

be the determinant with the element $\underset{a}{h}_i$ in the a'th row and the i'th column. Then it follows from (2.15), with $b = a$, that $\overset{a}{h}{}^i$ is equal to the conjugate minor of the element $\underset{a}{h}_i$ in this determinant divided by h. Therefore we also have

$$\underset{a}{h}_i \overset{a}{h}{}^k = \delta_i^k \tag{2.18}$$

and hence, by (2.11),

$$\underset{a}{h}_i \overset{a}{h}_k = g_{ik}, \quad \underset{a}{h}{}^i \overset{a}{h}{}^k = g^{ik}. \tag{2.19}$$

Further, since by (2.14)

$$\det\left\{\overset{a}{h}_k\right\} = -\det\left\{\underset{a}{h}_k\right\}, \tag{2.20}$$

we obtain from (2.19) for the determinant $g = \det\{g_{ik}\}$

$$g = -h^2, \quad |h| = \sqrt{-g}. \tag{2.21}$$

For a given tetrad field $\underset{a}{h}_i$ the metric tensor g_{ik} is determined by (2.19). However, for a given metric g_{ik} the tetrad field is by no means completely determined, since any rotation (Lorentz transformation) of the tetrads leads to a new set of tetrads $\underset{a}{\lambda}{}^i$ which also satisfy (2.19). In fact we have

$$\underset{a}{\lambda}{}^i = \underset{a}{\overset{b}{\Omega}}(x)\, \underset{b}{h}{}^i, \quad \underset{a}{\lambda}_i = \underset{a}{\overset{b}{\Omega}}(x)\, \underset{b}{h}_i, \tag{2.22}$$

where the "rotation coefficients" $\underset{a}{\overset{b}{\Omega}}(x)$ and the functions

$$\underset{b}{\overset{a}{\Omega}}(x) \equiv \varepsilon_{a)}\, \varepsilon_{b)}\, \underset{a}{\overset{b}{\Omega}}(x) \tag{2.23}$$

are scalars satisfying the orthogonality conditions

$$\underset{a}{\overset{c}{\Omega}} \underset{c}{\overset{b}{\Omega}} = \underset{a}{\overset{c}{\Omega}} \underset{c}{\overset{b}{\Omega}} = \delta_a^b. \tag{2.24}$$

(As a general rule the tetrad indices $a, b \ldots$ are lowered and raised by means of the constant matrices $\eta_{ab} = \eta^{ab}$, while the tensor indices $i, k, \ldots$ are raised and lowered by means of the metric tensor g_{ik}!).

Hence,

$$\underset{a}{\lambda}_i \overset{a}{\lambda}_k = \underset{a}{\overset{c}{\Omega}} \underset{b}{\overset{a}{\Omega}}\, \underset{c}{h}_i \overset{b}{h}_k = \underset{b}{h}_i \overset{b}{h}_k = g_{ik} \tag{2.25}$$

for arbitrary functions $\underset{a}{\overset{b}{\Omega}}(x)$ satisfying (2.24). Since the homogeneous Lorentz group is a 6-parametric group, the general solution $\underset{a}{h}_i(x)$ of (2.19) for given $g_{ik}(x)$ contains six arbitrary functions.

By means of (2.19) any function of the g_{ik}'s and their derivatives can be expressed in terms of the tetrads $\overset{}{\underset{a}{h}}{}^i$ and their derivatives. All expressions of this kind will of course be invariant under arbitrary rotations (2.22) of the tetrads. In particular this holds for the curvature scalar density which, as shown in Appendix A, can be written in the form

$$\mathfrak{R} \equiv \sqrt{-g}\, R = \hat{\mathfrak{L}} + \hat{\mathfrak{h}}, \tag{2.26}$$

where $\hat{\mathfrak{L}}$ is a homogeneous quadratic form in the first-order derivatives of the $\underset{a}{h}{}^i$, while $\hat{\mathfrak{h}}$ has the form of a divergence of a vector density. The explicit expression for $\hat{\mathfrak{L}}$ is

$$\hat{\mathfrak{L}} = |\, h\, | \left(\underset{a}{h}{}^r{}_{;s} \overset{a}{h}{}^s{}_{;r} - \underset{a}{h}{}^r{}_{;r} \overset{a}{h}{}^s{}_{;s} \right), \tag{2.27}$$

which is a scalar density under arbitrary space-time transformations in contrast to the usual Lagrangean $\mathfrak{L} = \mathfrak{L}(g^{ik}, g^{ik}{}_{,l})$, which has this property only under linear transformations. $\hat{\mathfrak{L}}$ is also invariant under rotations (2.22) with *constant* $\overset{b}{\underset{a}{\Omega}}$, but *not* under the general group of rotation with varying coefficients. This circumstance is of importance for the later development.

Since $\hat{\mathfrak{h}}$ has the form of an ordinary divergence, we may disregard it in the variational principle, i. e., for any variation $\delta \overset{a}{h}{}^k$ of the variables $\overset{a}{h}{}^k$ that vanishes at the surface of a region Σ in 4-space we have

$$\left.\begin{aligned} \delta \int_\Sigma \mathfrak{R}\, dx &= \delta \int_\Sigma \hat{\mathfrak{L}}\, dx \\ \text{or} \qquad \int_\Sigma \frac{\delta \mathfrak{R}}{\delta g^{ik}}\, \delta g^{ik}\, dx &= \int_\Sigma \frac{\delta \hat{\mathfrak{L}}}{\delta \overset{a}{h}{}^k}\, \delta \overset{a}{h}{}^k\, dx, \end{aligned}\right\} \tag{2.28}$$

where $\dfrac{\delta}{\delta g^{ik}}$ and $\dfrac{\delta}{\delta \overset{a}{h}{}^k}$ are the variational derivatives with respect to $g^{ik}(x)$ and $\overset{a}{h}{}^k(x)$.

Since

$$\frac{\delta \mathfrak{R}}{\delta g^{ik}} = \mathfrak{G}_{ik} = \sqrt{-g}\, G_{ik} = \mathfrak{G}_{ik} \tag{2.29}$$

(2.28) becomes, by means of (2.19),

$$2 \int_\Sigma \mathfrak{G}_{ik}\, \underset{a}{h}{}^i\, \delta \overset{a}{h}{}^k\, dx = \int_\Sigma \frac{\delta \hat{\mathfrak{L}}}{\delta \overset{a}{h}{}^k}\, \delta \overset{a}{h}{}^k\, dx. \tag{2.30}$$

Hence,

$$\mathfrak{G}_{ik}\,\underset{a}{h}{}^{i} = \frac{1}{2}\,\frac{\delta\hat{\mathfrak{L}}}{\delta\overset{a}{h}{}^{k}}$$

or

$$\mathfrak{G}_{ik} = \frac{1}{2}\,\overset{a}{h}_{i}\,\frac{\delta\hat{\mathfrak{L}}}{\delta\overset{a}{h}{}^{k}} = \frac{1}{2}\,\overset{a}{h}_{k}\,\frac{\delta\hat{\mathfrak{L}}}{\delta\overset{a}{h}{}^{i}} = -\varkappa\sqrt{-g}\,T_{ik} \tag{2.31}$$

by (2.7).

Now, the field equations (2.31) imply the covariant divergence relation

$$T^{k}_{i\,;\,k} \equiv \frac{1}{\sqrt{-g}}\,(\sqrt{-g}\,T^{k}_{i})_{,\,k} + \frac{1}{2}\,g^{lm}{}_{,\,i}\,T_{lm} = 0 \tag{2.32}$$

which may be written, by means of (2.31) and (2.19),

$$(\sqrt{-g}\,T^{k}_{i})_{,\,k} = -\underset{a}{h}{}^{l}\,\overset{a}{h}{}^{m}{}_{,\,i}\sqrt{-g}\,T_{lm} = \frac{1}{2\varkappa}\,\underset{a}{h}{}^{l}\,\overset{a}{h}{}^{m}{}_{,\,i}\,\overset{b}{h}_{l}\,\frac{\delta\hat{\mathfrak{L}}}{\delta\overset{b}{h}{}^{m}} = \frac{1}{2\varkappa}\,\overset{a}{h}{}^{m}{}_{,\,i}\,\frac{\delta\hat{\mathfrak{L}}}{\delta\overset{a}{h}{}^{m}}.$$

By introduction of the explicit expression for the variational derivatives we get

$$\left.\begin{aligned}(\sqrt{-g}\,T^{k}_{i})_{,\,k} &= \frac{\overset{a}{h}{}^{m}{}_{,\,i}}{2\varkappa}\left[\frac{\partial\hat{\mathfrak{L}}}{\partial\overset{a}{h}{}^{m}} - \left(\frac{\partial\hat{\mathfrak{L}}}{\partial\overset{a}{h}{}^{m}{}_{,\,k}}\right)_{,\,k}\right]\\ &= \frac{1}{2\varkappa}\left[\frac{\partial\hat{\mathfrak{L}}}{\partial\overset{a}{h}{}^{m}}\,\overset{a}{h}{}^{m}{}_{,\,i} + \frac{\partial\hat{\mathfrak{L}}}{\partial\overset{a}{h}{}^{m}{}_{,\,k}}\,\overset{a}{h}{}^{m}{}_{,\,i,\,k} - \left(\overset{a}{h}{}^{m}{}_{,\,i}\,\frac{\partial\hat{\mathfrak{L}}}{\partial\overset{a}{h}{}^{m}{}_{,\,k}}\right)_{,\,k}\right].\end{aligned}\right\} \tag{2.33}$$

Since the first two terms in the brackets in the last member of this equation combine into $\hat{\mathfrak{L}}_{i}$, (2.33) can be written

$$\mathsf{T}_{i}^{k}{}_{,\,k} = 0 \tag{2.34}$$

with

$$\mathsf{T}_{i}^{k} = \sqrt{-g}\,(T^{k}_{i} + t_{i}^{k}), \tag{2.35}$$

$$\sqrt{-g}\,t_{i}^{k} = \frac{1}{2\varkappa}\left[\frac{\partial\hat{\mathfrak{L}}}{\partial\overset{a}{h}{}^{l}{}_{,\,k}}\,\overset{a}{h}{}^{l}{}_{,\,i} - \delta^{k}_{i}\,\hat{\mathfrak{L}}\right]. \tag{2.36}$$

The equation (2.34), which is equivalent to (2.32), shows that the complex T_{i}^{k} defined by (2.35-36) satisfies the condition 2.

The relations (2.34-36) could also have been obtained by application of the method of infinitesimal transformations (see for instance [3] and Appendix A) to the true scalar density $\hat{\mathfrak{L}}/2\varkappa$. By the same method one finds moreover that the complex T_{i}^{k} is derivable from a superpotential $\mathfrak{U}_{i}^{kl}$.

$$\mathsf{T}_i^{\ k} = \mathfrak{U}_i^{\ kl}{}_{,\,l} \tag{2.37}$$

with

$$\mathfrak{U}_i^{\ kl} = \frac{1}{2\varkappa}\frac{\partial \hat{\mathfrak{L}}}{\partial \overset{a}{h}{}^i{}_{,\,l}}\, \underset{a}{h}{}^k = -\mathfrak{U}_i^{\ lk}. \tag{2.38}$$

As shown in Appendix A, the explicit expressions (2.36) and (2.38) for t_i^k and $\mathfrak{U}_i^{\ kl}$ with $\hat{\mathfrak{L}}$ given by (2.27) are

$$t_i^k = \frac{1}{\varkappa}\left\{ \overset{a}{h}{}^k{}_{;\,l}\, \underset{a}{h}{}^l{}_{,\,i} - \overset{a}{h}{}^r{}_{;\,r}\, \underset{a}{h}{}^k{}_{,\,i} + \overset{a}{h}{}_r\, \underset{a}{h}{}^r{}_{,\,i}\, \overset{b}{h}{}^k\, \underset{b}{h}{}^s{}_{;\,s} - \frac{1}{2\,|\,h\,|}\, \delta_i^k\, \hat{\mathfrak{L}} \right\} \tag{2.39}$$

$$\mathfrak{U}_i^{\ kl} = -\mathfrak{U}_i^{\ lk} = \frac{|\,h\,|}{\varkappa}\left[\underset{a}{h}{}^k\, \overset{a}{h}{}^l{}_{;\,i} + \left(\delta_i^k\, \overset{a}{h}{}^l - \delta_i^l\, \overset{a}{h}{}^k\right) \underset{a}{h}{}^s{}_{;\,s} \right]. \tag{2.40}$$

Note that

$$\underset{a}{h}{}^k\, \overset{a}{h}{}^l{}_{;\,i} = -\underset{a}{h}{}^k{}_{;\,i}\, \overset{a}{h}{}^l \tag{2.41}$$

on account of 2.19 and the identity $g^{kl}{}_{;\,i} \equiv 0$.

In contrast to the superpotential $h_i^{\ kl}$ of the Einstein expression $\Theta_i^{\ k}$, $\mathfrak{U}_i^{\ kl}$ is seen to be a true tensor density of rank 3, antisymmetric in the indices k and l. This means that $\mathfrak{U}_4^{\ kl}$ is transformed as an antisymmetric tensor density of rank 2 under the group of purely spatial transformations (2.3), and consequently $\mathsf{T}^k = \mathfrak{U}_4^{\ kl}{}_{,\,l}$ will be transformed as a vector density under this group of transformations. Thus the complex $\mathsf{T}_i^{\ k}$ also satisfies the condition 4.

Now, consider an arbitrary static, spherically symmetric system in a system of "isotropic" coordinates. Here the metric tensor is of the form

$$\left.\begin{aligned} g_{ik} &= g_{ii)}\, \delta_{ik}, \qquad g^{ik} = \frac{\delta^{ik}}{g_{ii)}} \\ g_{ii} &= \left\{a(r),\, a(r),\, a(r),\, -b(r)\right\}, \end{aligned}\right\} \tag{2.42}$$

where $a(r)$ and $b(r)$ are functions of $r = \sqrt{(x^1)^2 + (x^2)^2 + (x^2)^3}$ only. In that case a possible and in fact very natural choice of tetrads is

$$\left.\begin{aligned} \underset{a}{h}{}^i &= \frac{1}{\sqrt{|\,g_{aa}\,|}}\, \delta_a^i, \qquad \underset{a}{h}{}_i = g_{ik}\, \underset{a}{h}{}^k = \frac{g_{ii)}\, \delta_{ai}}{\sqrt{|\,g_{aa}\,|}} = \varepsilon_{a)} \sqrt{|\,g_{aa}\,|}\, \delta_{ai} \\ \overset{a}{h}{}^i &= \frac{\varepsilon_{a)}\, \delta_a^i}{\sqrt{|\,g_{aa}\,|}}, \qquad \overset{a}{h}{}_i = \sqrt{|\,g_{aa}\,|}\, \delta_{ai}. \end{aligned}\right\} \tag{2.43}$$

This choice is easily seen to be in accordance with (2.19) and (2.42). In Appendix B the superpotentials $\mathfrak{U}_i^{\ kl}$ and $h_i^{\ kl}$ defined by 2.40 and 2.6 are calculated for the tetrads (2.43) and the metric (2.42), and are found to be equal. In fact we obtain

$$\mathfrak{U}_i^{kl} = h^{kl} = \frac{\sqrt{ab}}{\varkappa}\frac{d}{dr}\left[ln\frac{a\sqrt{b}}{|g_{ii}|}\right]\cdot(\delta_i^k n^l - \delta_i^l n^k) \tag{2.44}$$

with

$$n^k = \left\{\frac{x^1}{r}, \frac{x^2}{r}, \frac{x^3}{r}, 0\right\}. \tag{2.45}$$

Now, for any closed system the metric is given *asymptotically*, i. e. for $r \to \infty$, by (2.42) with

$$a(r) \sim 1 + \alpha/r, \quad b(r) \sim 1 - \alpha/r, \tag{2.46}$$

where the constant α is connected with the total gravitational mass M_0 of the system by

$$\alpha = 2\,k\,M_0/c^2 = \frac{\varkappa c^2}{4\pi} M_0 \tag{2.47}$$

(k is the Newtonian gravitational constant). Therefore the complex $\mathsf{T}_i^k = \mathfrak{U}_i^{kl}{}_{,l}$ will satisfy condition 3 in the same way as EINSTEIN's expression Θ_i^k. In this connection it is essential that t_i^k is a homogeneous quadratic function of the $\underset{a}{h}{}^i{}_{,k}$, which are of the order $O\left(\frac{1}{r^2}\right)$ for $r \to \infty$. Thus the relations (2.8) are valid also for t_i^k and T_i^k:

$$t_i^k \sim O\left(\frac{1}{r^4}\right), \quad \mathsf{T}_i^k \sim O\left(\frac{1}{r^4}\right) \tag{2.48}$$

for $r \to \infty$.

As we have seen above, the superpotentials $\mathfrak{U}_i^{kl}$ and h_i^{kl} are identical in the static, spherically symmetric case if we use isotropic coordinates. Since h_i^{kl} is an affine tensor density, this identity will remain true in any system of coordinates obtained from the isotropic system by a *linear* transformation. In all such systems we therefore have

$$\mathsf{T}_i^k = \mathfrak{U}_i^{kl}{}_{,l} = h_i^{kl}{}_{,l} = \Theta_i^k. \tag{2.49}$$

However, the identity (2.49) does not hold in more general systems of coordinates, since $\mathfrak{U}_i^{kl}$ is a true tensor density while h_i^{kl} is not. In particular, T_4^4 is a scalar density under the group of purely spatial transformations (2.3), while Θ_4^4 changes in an unphysical way under these transformations. This means that the quantity (2.4) may be interpreted consistently as the energy content of the finite volume V of 3-space. In this respect the complex T_i^k, which satisfies all the conditions 1.–4., is superior to the original expression of EINSTEIN, which gives the "correct" energy distribution only for the restricted class of coordinate systems in which (2.49) holds. In particular, $\Theta_i^k \neq \mathsf{T}_i^k$ in the system of "harmonic" coordinates so strongly advocated by V. FOCK[(9)], since this system is obtained from the isotropic system by a transformation of the type (2.3).

Although the complex T_i^k, defined by (2.37–40) together with (2.43), seems to furnish a satisfactory description of a spherically symmetric system, the problem of the energy distribution cannot yet be regarded as solved. In fact, as emphasized in the last section of [4], we would get another complex satisfying the conditions 1.–4. by using a tetrad field $\underset{a}{\lambda}^i$ obtained from the tetrads $\underset{a}{h}^i$ in (2.43) by rotations of the type (2.22) with arbitrary functions $\underset{a}{\overset{b}{\Omega}}(x)$, which tend sufficiently rapidly to constant values for $r \to \infty$, but are otherwise arbitrary. For, as emphasized earlier, $\hat{\mathfrak{L}}$ and hence T_i^k are *not* invariant under such general rotations of the tetrads, but only under rotations with constant coefficients $\underset{a}{\overset{b}{\Omega}}$. In order to obtain a unique expression for the energy-momentum complex we obviously need a set of conditions that fix the relative orientation of the tetrads in different space-time points, and we may expect that these yet unknown conditions are satisfied by the tetrads (2.43) in the special case of a spherically symmetric system, and that these tetrads should be uniquely determined by the supplementary conditions.

3. On the Conditions Defining the Relative Orientation of the Tetrads

The considerations of the previous section lead to the idea that not all tetrads satisfying (2.19) can be used in the expression for the energy-momentum complex, but only a certain class of tetrads with a fixed relative orientation. This means that the "correct" tetrads $\underset{a}{h}^i(x)$ must satisfy supplementary equations which, together with the field equations (2.31), determine the $\underset{a}{h}^i(x)$ uniquely apart from an overall rotation (2.23) with *constant* coefficients $\underset{a}{\overset{b}{\Omega}}$. These supplementary equations must be generally covariant under space-time transformations and invariant under constant rotations of the tetrads, and, since the general solution $\underset{a}{h}^i(x)$ of (2.19) for given $g_{ik}(x)$ contains six arbitrary functions, they must be of the form

$$\varphi_{ik} = 0, \tag{3.1}$$

where φ_{ik} is an antisymmetrical tensor of rank 2 depending on the tetrad fields $\underset{a}{h}^i(x)$ and their derivatives in such a way that the index a occurs as a dummy index only. (The alternative possibility that the supplementary equations are six scalar equations or two scalar and one vector equation is ruled out since it is impossible to construct a sufficient number of quantities of that type from the $\underset{a}{h}^i$ and their derivatives.) The simplest assumption is now that φ_{ik} is a local quantity, which means that the six equations (3.1) are differential equations. Therefore, the relative orientation of the tetrads $\underset{a}{h}^i$ can be fixed by the equations (3.1) only if these are supplemented by suitable initial or boundary conditions.

Let us first settle the question of boundary conditions. In the present paper we shall not be interested in cosmological problems, i. e. we shall consider large but limited parts of the universe, where, for large spatial distances from the matter, space-time may be regarded as flat. Now, consider first the trivial case of no matter at all. Here $T_i^k = 0$ everywhere, and also the Riemann curvature tensor will be zero in all the points of space-time under consideration. In this case we may further assume that t_i^k in (2.35–36) and (2.39) has to be zero everywhere, which means that the vector fields $\underset{a}{h}^i$ must be stationary, i. e.

$$\underset{a}{h}^i{}_{;k} = 0 \tag{3.2}$$

in all points. In a pseudo-Cartesian system of coordinates, where the metric is given by

$$g_{ik} = \eta_{ik}, \tag{3.3}$$

the components of the tetrads are then constant, and we may choose

$$\underset{a}{h}^i = \delta_a^i. \tag{3.4}$$

Let us now consider an arbitrary "insular" system, where the matter is limited to a finite part of space. This means that T_i^k is zero outside a certain "world tube" with a finite spatial cross section. Here we may assume that space-time is asymptotically flat for large values of the spatial distance r. This does *not* necessarily mean that the system is closed, i. e. that the metric is given asymptotically by the Schwarzschild solution. It may very well emit "gravitational waves". But we exclude the emission of electromagnetic waves since it would make the world tube of matter extend to "spatial infinity".

As space-time is asymptotically flat, it is possible (but by no means necessary) to introduce asymptotically pseudo-Cartesian coordinates in which (3.3) holds for $r \to \infty$. The first boundary condition for the $\underset{a}{h}^i(x)$ then reads

A.
$$\underset{a}{h}^i(x) \to \delta_a^i \quad \text{for} \quad r \to \infty. \tag{3.5}$$

Further we shall require that

B. $\underset{a}{h}^i(x) - \delta_a^i$ shows *the same asymptotic behaviour as the metric quantities* $g_{ik} - \eta_{ik}$ *for an insular system. This means that, if* $\psi(x)$ *is any of these quantities,* ψ *must satisfy the condition of outward radiation*(10)

$$\lim\left(\frac{\partial\,(r\psi)}{\partial\,r} + \frac{1}{c}\frac{\partial\,(r\psi)}{\partial\,t}\right) = 0 \tag{3.6}$$

for $r \to \infty$ *and for all values of* $t_0 = t + r/c$ *in an arbitrary fixed interval. Moreover,* $\psi(x)$ *and its first-order derivatives* $\psi_{,i}$ *must be everywhere bounded and must go to zero at least as* $1/r$ *for* $r \to \infty$.

We now return to the question of the form of the differential equations (3.1), and we shall assume that φ_{ik} does not contain derivatives of the $\underset{a}{h}^i(x)$ of orders higher than the second.

Among the tensors that can be constructed from the $\underset{a}{h}^i$'s and their first-order derivatives there is one with a simple geometrical meaning, viz.

$$\gamma_{ikl} = \overset{a}{h}_i \, \underset{a}{h}_{k;\,l} = -\gamma_{kil}. \tag{3.7}$$

Its components along the directions of the tetrad are the "Ricci rotation coefficients"

$$\underset{abc}{\gamma} = -\underset{bac}{\gamma} = \gamma_{ikl} \, \underset{a}{h}^i \, \underset{b}{h}^k \, \underset{c}{h}^l = \underset{a}{h}_{k;\,l} \, \underset{b}{h}^k \, \underset{c}{h}^l. \tag{3.8}$$

Obviously $\underset{abc}{\gamma}$ is transformed as a tensor under *constant* rotations of the tetrad. Therefore it is called a local tetrad tensor. From the expression (3.8) it follows that

$$\underset{abc}{\gamma} = \frac{D\underset{a}{h}^k}{d\,|\underset{c}{s}|} \, \underset{b}{h}_k, \tag{3.9}$$

where $\frac{D}{d\,|\underset{c}{s}|}$ is the covariant derivative with respect to the invariant measure $|\underset{c}{s}|$ along a curve that has the unit vector $\underset{c}{h}^l$ as a tangent.

By contraction of (3.7) we arrive at the vector

$$\Phi_k = \gamma^i{}_{ki} = -\gamma_k{}^i{}_i = \overset{a}{h}^i \, \underset{a}{h}_{k;\,i}. \tag{3.10}$$

From (3.7) and (3.10) we at once obtain, by (2.15),

$$\underset{a}{h}_{k;\,l} = \underset{a}{h}^i \gamma_{ikl}, \qquad \underset{a}{h}^r{}_{;\,r} = -\underset{a}{h}^r \Phi_r. \tag{3.11}$$

Insertion of these expressions into (2.27), (2.38–39) gives by means of (2.19), (2.21)

$$\hat{\mathfrak{L}} = |h| \left[\gamma_{rst} \gamma^{tsr} - \Phi_r \Phi^r\right] \tag{3.12}$$

$$\mathfrak{U}_i{}^{kl} = \frac{|h|}{\varkappa} \left[\gamma^{kl}{}_i - \delta_i^k \Phi^l + \delta_i^l \Phi^k\right] \tag{3.13}$$

$$t_i^k = \frac{1}{\varkappa}\left[\gamma^{km}{}_l \Delta^l_{mi} - \Phi^l \Delta^k{}_{li} + \Delta^l{}_{li} \Phi^k - \frac{1}{2\,|h|} \delta_i^k \hat{\mathfrak{L}}\right]. \tag{3.14}$$

Here we have introduced the notation

$$\Delta^i{}_{kl} = \overset{a}{h}^i \, \underset{a}{h}_{k,\,l}. \tag{3.15}$$

This quantity is *not* a tensor. Therefore, while $\hat{\mathfrak{L}}$ and $\mathfrak{U}_i^{kl}$ are true tensor densities, $t_i^{\,k}$ is not a tensor. Nevertheless, as we have seen in section 2, $t_4^{\,k}$, which determines the energy density and the energy current of the gravitational field, is transformed as a vector under the group of purely spatial transformations (2.3). This follows at once from the fact that $\mathsf{T}_i^{\,k}$ as well as the matter tensor density $\sqrt{-g}\,T_i^{\,k}$ satisfy the condition 4. on p. 6.

From γ_{ikl} and Φ^k we can construct two independent antisymmetric tensors of rank 2 containing only first-order derivatives, viz.

$$\zeta_{ik} = \gamma_{ikl}\,\Phi^l \tag{3.16}$$

$$\tau_{ik} = \Phi^l(\gamma_{lik} - \gamma_{lki}). \tag{3.17}$$

They are both homogeneous quadratic functions of the first-order derivatives of the tetrad functions.

Similarly, we can form two independent antisymmetrial tensors of rank 2 containing also second-order derivatives, viz.

$$\xi_{ik} = \gamma_{ik}{}^{l}{}_{;\,l} + \gamma_{ikl}\Phi^l \tag{3.18}$$

$$\eta_{ik} = \Phi_{k,\,i} - \Phi_{i,\,k}. \tag{3.19}$$

The second one is the curl of the vector Φ_i, and ξ_{ik} in (3.18) has the following simple geometrical meaning. The tetrad components of this tensor are the six invariants

$$\underset{ab}{\xi} = \xi_{ik}\,\underset{a}{h^i}\,\underset{b}{h^k} = \underset{a}{h^i}\,\underset{b}{h^k}\,(\gamma_{ik}{}^{l}{}_{;\,l} + \gamma_{ikl}\Phi^l) \tag{3.20}$$

which are easily seen to be equal to the "local divergence" of the tetrad tensor $\underset{ab}{\gamma^{c}}$, i. e.

$$\underset{ab}{\xi} = \frac{d\underset{ab}{\gamma^{c}}}{d\,|\underset{c}{s}|}, \qquad \text{(summation over } c!) \tag{3.21}$$

where $|\underset{c}{s}|$ is again the measure along a curve that has $\underset{c}{h^i}$ as a tangent. In fact, we have from the definition of $\underset{ab}{\gamma^{c}}$, i. e.

$$\underset{ab}{\gamma^{c}} = \underset{a}{h^i}\,\underset{b}{h^k}\,\overset{c}{h^l}\,\gamma_{ikl} \tag{3.22}$$

and, by means of (2.19), (3.10-11),

$$\frac{d\underset{ab}{\gamma^{c}}}{d\,|\underset{c}{s}|} = \underset{ab}{\gamma^{c}}{}_{;\,m}\,\underset{c}{h^m} = \underset{a}{h^i}\,\underset{b}{h^k}\,\overset{c}{h^l}\,\gamma_{ikl;\,m}\,\underset{c}{h^m} + \gamma_{ikl}\,\underset{c}{h^m}\,(\underset{a}{h^i}{}_{;\,m}\,\underset{b}{h^k}\,\overset{c}{h^l} + \underset{a}{h^i}\,\underset{b}{h^k}{}_{;\,m}\,\overset{c}{h^l} + \underset{a}{h^i}\,\underset{b}{h^k}\,\overset{c}{h^l}{}_{;\,m})$$

$$= \underset{a}{h^i}\,\underset{b}{h^k}\,(\gamma_{ik}{}^{l}{}_{;\,l} + \gamma_{ikl}\Phi^l) = \underset{ab}{\xi}$$

on account of (3.20).

Now, the most general tensor of the type of φ_{ik} has the form of a linear combination of the tensors (3.16-19), i. e.

$$\varphi_{ik} = \alpha\,\xi_{ik} + \beta\,\eta_{ik} + \gamma\zeta_{ik} + \delta\tau_{ik} \tag{3.23}$$

with arbitrary constants $\alpha, \beta, \gamma, \delta$. In the supplementary equations (3.1) only the ratios between these constants are of importance. In order to fix these ratios we obviously have to use criteria other than mere covariance. The first criterion one would think of is that the tetrads (2.43) in the spherically symmetric case should satisfy (3.1). However, as shown in Appendix *B*, all four quantities (3.16-19) are separately zero with $\underset{a}{h^i}$ given by (2.43) so that this requirement is no help in removing the arbitrariness involved in (3.1), (3.23). A more useful criterion is that the differential equations (3.1), together with the boundary conditions A, B, should uniquely determine the relative orientation of the tetrads. This question will be treated in the following section.

4. Investigation of the Supplementary Conditions in the Case of Weak Fields

The question of the uniqueness of the solutions of (2.31), (3.1) and A, B in general requires an investigation of the properties of non-linear differential equations, which are difficult to handle mathematically. For this reason we shall here consider only the case of weak gravitational fields, where the equations involved become linear.

By definition the gravitational field of an insular system is called weak if it is possible to introduce a system of coordinates (x^i) in which the metric is of the form

$$g_{ik} = \eta_{ik} + y_{ik}(x), \tag{4.1}$$

where the quantities $y_{ik}(x) = y_{ki}(x)$ are small functions of the first order satisfying the boundary condition B. Then

$$g^{ik} = \eta^{ik} - y^{ik}, \tag{4.2}$$

where

$$y^{ik} = \varepsilon_{i)}\,\varepsilon_{k)}\,y_{ik} \tag{4.3}$$

with the numbers ε_i given by (2.13). We shall also use the definitions

$$y_i^k = \varepsilon_{k)}\,y_{ik}, \qquad y = y_i^i. \tag{4.4}$$

In this case, the equations (2.19) allow us to choose (in an infinite number of ways) tetrads $\underset{a}{\lambda_i}$ of the form

$$\underset{a}{\lambda_i} = \eta_{ai} + \underset{a}{\mu_i}(x), \tag{4.5}$$

where the quantities $\underset{a}{\mu_i}(x)$ are small functions of the first order, satisfying the boundary condition *B*. Then,

$$\overset{a}{\lambda}_i = \delta^a_i + \overset{a}{\mu}_i(x), \qquad \overset{a}{\mu}_i = \varepsilon_{a)} \underset{a}{\mu}_i \tag{4.6}$$

and (2.19) gives to the first order, by (4.1),

$$(\eta_{ai} + \underset{a}{\mu}_i)(\delta^a_k + \overset{a}{\mu}_k) = \eta_{ik} + \underset{i}{\mu}_k + \underset{k}{\mu}_i = \eta_{ik} + y_{ik},$$

i. e.,

$$y_{ik} = \underset{i}{\mu}_k + \underset{k}{\mu}_i. \tag{4.7}$$

Further we have by (4.2-5)

$$\underset{a}{\lambda}^i = g^{ik}\underset{a}{\lambda}_k = (\eta^{ik} - y^{ik})(\eta_{ak} + \underset{a}{\mu}_k) = \delta^i_a - y^i_a + \underset{a}{\mu}^i \tag{4.8}$$

with

$$\underset{a}{\mu}^i = \eta^{ik}\underset{a}{\mu}_k = \varepsilon_{i)}\underset{a}{\mu}_i. \tag{4.9}$$

(It is a general rule that for first-order quantities, like $\underset{a}{\mu}^i$, the tetrad indices $a, b, \ldots$ and the tensor indices $i, k, \ldots$ are raised and lowered by means of the same numbers $\varepsilon_a, \varepsilon_b, \ldots$ and $\varepsilon_i, \varepsilon_k, \ldots$).

Hence, by (4.3-4) and (4.7),

$$y^{ik} = \overset{i}{\mu}^k + \overset{k}{\mu}^i, \quad y^k_i = \underset{i}{\mu}^k + \overset{k}{\mu}_i \tag{4.10}$$

and by (4.8),

$$\underset{a}{\lambda}^i = \delta^i_a - \overset{i}{\mu}_a, \quad \overset{a}{\lambda}^i = \eta^{ai} - \overset{i}{\mu}^a. \tag{4.11}$$

Now, the "right" tetrads $\underset{a}{h}^i$, satisfying the supplementary equations (3.1) and the boundary conditions A, B, must in this case be obtainable from the $\underset{a}{\lambda}^i$ by an *infinitesimal* rotation

$$\underset{a}{h}_i = \underset{a}{\overset{b}{\Omega}}(x)\underset{b}{\lambda}_i \tag{4.12}$$

$$\underset{a}{\Omega}^b = \delta^b_a + \underset{a}{\omega}^b(x) \tag{4.13}$$

where the small first-order scalar functions

$$\underset{ab}{\omega}(x) = \varepsilon_{b)}\underset{a}{\omega}^b(x) = -\underset{ba}{\omega}(x) \tag{4.14}$$

are antisymmetric in a and b and satisfy the boundary condition B. The symmetry property (4.14) is a consequence of the orthogonality relations (2.24) and clearly displays the fact that there are just six arbitrary functions $\underset{ab}{\omega}(x)$ involved, which have to be determined by means of the supplementary equations.

Now we have from (4.5), (4.12-14)

$$\underset{a}{h}_i = (\delta^b_a + \underset{a}{\omega}^b)(\eta_{bi} + \underset{b}{\mu}_i) = \eta_{ai} + \underset{ai}{\omega} + \underset{a}{\mu}_i. \tag{4.15}$$

Further, by means of (4.7), we obtain up to the first order for the Christoffel symbols

$$\Gamma^{i}_{kl} = \frac{\varepsilon_{i)}}{2}\left[(\underset{i}{\mu_k} + \underset{k}{\mu_i}),_l + y_{il,k} - y_{kl,i} \right]. \tag{4.16}$$

Thus, by (3.7) and (4.15-16), to the same order,

$$\gamma_{ikl} = \overset{a}{h_i} \underset{a}{h_{k;l}} = \underset{i}{h_{k,l}} - \underset{i}{h_r} \Gamma^{r}_{kl} = (\underset{ik}{\omega} + \underset{i}{\mu_k}),_l - \frac{1}{2}\left[(\underset{i}{\mu_k} + \underset{k}{\mu_i}),_l + y_{il,k} - y_{kl,i} \right]$$

or

$$\gamma_{ikl} = \nu_{ik,l} + \frac{1}{2}(y_{kl,i} - y_{il,k}), \tag{4.17}$$

where we have put

$$\nu_{ik} = \underset{ik}{\omega} + \frac{1}{2}(\underset{i}{\mu_k} - \underset{k}{\mu_i}) = -\nu_{ki}. \tag{4.18}$$

Obviously ν_{ik}, like $\underset{ik}{\omega}$ and $\underset{i}{\mu_k}$, has to satisfy the boundary condition B.

For the vector Φ_k we get by (4.17)

$$\Phi_k = \gamma^{i}{}_{ki} = \varepsilon_i \gamma_{iki} = \nu^{i}{}_{k,i} + \frac{1}{2}(y^{i}_{k,i} - y,_k). \tag{4.19}$$

Since γ_{ikl} and Φ_k are small of the first order, the tensors ζ_{ik} and τ_{ik} defined by (3.16-17) are small of the second order, and they have to be consistently neglected in the expression (3.23) for φ_{ik}. In the same approximation the tensors ξ_{ik} and η_{ik} in (3.18-19) become

$$\xi_{ik} = \gamma_{ik}{}^{l}{}_{,l} = \Box\, \nu_{ik} + \frac{1}{2}(y^{l}_{k,i} - y^{l}_{i,k}),_l \tag{4.20}$$

$$\eta_{ik} = (\nu^{l}{}_{k,i} - \nu^{l}{}_{i,k}),_l + \frac{1}{2}(y^{l}_{k,i} - y^{l}_{i,k}),_l \tag{4.21}$$

and

$$\varphi_{ik} = \alpha\, \xi_{ik} + \beta\, \eta_{ik}. \tag{4.22}$$

Here,

$$\Box = \varepsilon_l \frac{\partial^2}{\partial x^{l\,2}} \tag{4.23}$$

is the usual special-relativity expression for d'Alembert's operator.

The expressions (4.20-22) are further simplified if we use a system of *harmonic* coordinates in which the de Donder condition is satisfied:

$$\left(y^{k}_{i} - \frac{1}{2}\delta^{k}_{i} y \right),_k = y^{k}_{i,k} - \frac{1}{2} y,_i = 0. \tag{4.24}$$

In harmonic coordinates we have simply

$$\xi_{ik} = \Box\, \nu_{ik}, \quad \eta_{ik} = \nu^{l}{}_{k,l,i} - \nu^{l}{}_{i,l,k}, \tag{4.25}$$

and the supplementary equations (3.1) take the form

$$\varphi_{ik} = \alpha \Box\, \nu_{ik} + \beta (\nu^{l}{}_{k,l,i} - \nu^{l}{}_{i,l,k}). \tag{4.26}$$

These equations, together with the boundary condition B, should determine ν_{ik} and ω_{ik} uniquely.

Therefore we see at once that α in (4.26) cannot be chosen equal to zero; for the equation

$$\eta_{ik} = \nu^{l}{}_{k,l,i} - \nu^{l}{}_{i,l,k} = 0 \tag{4.27}$$

would be satisfied by any ν_{ik} for which

$$\nu_{i}{}^{k}{}_{,k} = -\nu^{k}{}_{i,k} = 0, \tag{4.28}$$

and (4.28) holds for any ν^{ik} of the form

$$\nu^{ik} = \frac{1}{2}\delta^{iklm}(\varphi_{m,l} - \varphi_{l,m}), \tag{4.29}$$

where δ^{iklm} is the Levy-Civita symbol and the $\varphi_m(x)$ are arbitrary functions of the coordinates. Thus, the equations (4.27) are *not* suitable for the fixation of the relative orientation of the tetrads. It is essential to have a non-vanishing admixture of ξ_{ik} in the expression for φ_{ik}. On the other hand, if $\alpha \neq 0$, we have from (4.26), since ν_{ik} is antisymmetrical in i and k,

$$\varphi^{ik}{}_{,k} = (\alpha + \beta) \Box\, \nu^{ik}{}_{,k} = 0. \tag{4.30}$$

Now, it is easily seen that, if all the functions ν^{ik} satisfy the boundary condition B, also the functions $\nu^{ik}{}_{,k}$ will satisfy that condition. Therefore, if we assume

$$\beta \neq -\alpha, \quad \alpha \neq 0, \tag{4.31}$$

it follows from (4.30) that the $\nu^{ik}{}_{,k}$ are solutions of the wave equation

$$\Box\, \Psi = 0, \tag{4.32}$$

satisfying the boundary condition B. However, as emphasized by V. Fock[(10)], the only solution of this kind is

$$\nu^{ik}{}_{,k} = 0. \tag{4.33}$$

If we use (4.33) in (4.26), we see that also ν_{ik} is a solution of the wave equation (4.32) satisfying the boundary condition B. Therefore, we also find

$$\nu_{ik} = 0 \tag{4.34}$$

or, by (4.18),

$$\underset{ik}{\omega} = -\frac{1}{2}\left(\underset{i}{\mu}_k - \underset{k}{\mu}_i\right). \tag{4.35}$$

The result of this investigation is that, for weak fields at least, the relative orientation of the tetrads is uniquely determined by (3.1) and the boundary conditions A, B, provided that the constants in (3.23) are chosen in accordance with (4.31). The unique solution (4.35) was seen to be independent of the particular values of the constants; for reasons of simplicity we shall therefore choose

$$\beta = \gamma = \delta = 0, \quad \alpha \neq 0 \tag{4.36}$$

so that (3.1) is reduced to

$$\xi_{ik} = 0. \tag{4.37}$$

However, we shall keep in mind that a more complicated expression for φ_{ik} is possible if the later development should make it necessary to use it.

From (4.15), (4.35) and (4.7) we now get for the components of the right tetrads in harmonic coordinates

$$\underset{a}{h}_i = \eta_{ai} + \frac{1}{2}\left(\underset{i}{\mu}_a + \underset{a}{\mu}_i\right) = \eta_{ai} + \frac{1}{2}\, y_{ai}, \tag{4.38}$$

$$\overset{a}{h}{}^i = g^{ik}\,\overset{a}{h}_k = \left(\eta^{ik} - y^{ik}\right)\left(\delta^a_k + \frac{1}{2}\, y^a_k\right) = \eta^{ai} - \frac{1}{2}\, y^{ai}. \tag{4.39}$$

Thus, in harmonic coordinates and for weak fields the tetrads $\underset{a}{h}_i$ are simple functions of the metric tensor g_{ik}. In other systems of coordinates obtained from the harmonic system by an infinitesimal transformation the connection between $\underset{a}{h}_i$ and g_{ik} is generally not so simple. As shown in Appendix C, we have in general, instead of (4.38),

$$\underset{a}{h}_i = \eta_{ai} + \frac{1}{2}\left(y_{ai} + \xi_{a,i} - \xi_{i,a}\right), \tag{4.40}$$

where

$$\left.\begin{aligned} \xi_i &= \varepsilon_{i)}\, \xi^i(x) = -\frac{1}{4\pi}\int \frac{\delta\left(x^4 - x'^4 - R\right)}{R}\left(y^l_{i,l}(x') - \frac{1}{2}\, y_{,i}(x')\right) dx'^1\, dx'^2\, dx'^3\, dx'^4 \\ R &= \sqrt{\textstyle\sum_{\iota}\left(x^{\iota} - x'^{\iota}\right)^2} \end{aligned}\right\} \tag{4.41}$$

and the $\xi^i(x)$ are the transformation functions which connect the coordinates (x^i) with the harmonic coordinates $(x^i_{\text{harm.}})$ by the relation

$$x^i_{\text{harm.}} = x^i + \xi^i(x). \tag{4.42}$$

Thus, for weak fields, the harmonic coordinates are distinguished by the fact that $\underset{a}{h}_i$ in these coordinates are symmetrical in a and i. In other systems of coordinates, $\underset{a}{h}_i$

will in general also have an antisymmetrical part which has the form of a curl of a vector and which in general is a non-local function (a functional) of the metric tensor. However, in principle all systems of coordinates are equally allowed.

In harmonic coordinates and for weak fields the field equations (2.7) are reduced in the first-order approximation to

$$\frac{1}{2}\square\left(y_i^k-\frac{1}{2}\,\delta_i^k\,y\right)=-\varkappa\,T_i^k. \tag{4.43}$$

The solution of this equation satisfying the boundary condition B is given by the usual retarded integral

$$y_{ik}(x)=\frac{\varkappa}{2\pi}\int\left[T_{ik}(x')-\frac{1}{2}\,\eta_{ik}\,T_l^l(x')\right]\frac{\delta(x^4-x'^4-R)}{R}\,dx'. \tag{4.44}$$

In the trivial case of a completely empty space we obviously have, by (4.44) and (4.38)

$$y_{ik}=0,\quad \underset{a}{h}{}^i=\delta_a^i \tag{4.45}$$

which in arbitrary coordinates reads

$$\underset{a}{h}{}^i{}_{;k}=0. \tag{4.46}$$

Thus, the equations (3.2) and hence

$$\mathsf{T}_i^{\,k}=\sqrt{-g}\,t_i^{\,k}=0 \tag{4.47}$$

are consequences of the supplementary equations (4.37) and the boundary conditions A, B in the case of a completely empty space.

If there is matter present, we have in harmonic coordinates, according to (4.17-19) (4.24) and (4.34),

$$\gamma_{ikl}=\frac{1}{2}\,(y_{kl,i}-y_{il,k}),\quad \Phi_k=-\frac{1}{4}\,y_{,k}. \tag{4.48}$$

Thus, to the first order, the superpotential (3.13) becomes

$$\mathfrak{U}_i^{\,kl}=\frac{1}{2\varkappa}\left[y_i^{l,k}-y_i^{k,l}+\frac{1}{2}\,\delta_i^k\,y^{,l}-\frac{1}{2}\,\delta_i^l\,y^{,k}\right] \tag{4.49}$$

with

$$A^{,l}=\eta^{lm}A_{,m}=\varepsilon_{l)}A_{,l}. \tag{4.50}$$

In Appendic C the superpotential $h_i^{\,kl}$ is calculated to the same order of approximation in harmonic coordinates, and it is found that also $h_i^{\,kl}$ is given by the expression (4.49). Thus, in this approximation the energy-momentum complex $\mathsf{T}_i^{\,k}$ is identical with Einstein's expression $\Theta_{i\cdot}^{\,k}$. In fact both of them are equal to the matter tensor, for we have, on account of (4.24),

$$\left.\begin{aligned} \mathsf{T}_i^{\ k} &= \mathfrak{U}_i^{\ kl}{}_{,l} = h_i^{\ kl}{}_{,l} = \Theta_i^{\ k} \\ &= \frac{1}{2\varkappa}\left[y^l_{i,l}{}^{,k} - \Box\, y^k_i + \frac{1}{2}\,\delta^k_l\,\Box\, y - \frac{1}{2}\,y^{,k}{}_{,i}\right] \\ &= -\frac{1}{2\varkappa}\,\Box\left(y^k_i - \frac{1}{2}\,\delta^k_i\,y\right) = -\frac{1}{\varkappa}\,G^k_i = T^k_i. \end{aligned}\right\} \quad (4.51)$$

This result is in accordance with (2.35) since $t_i^{\ k}$ in (2.36), as well as the corresponding quantity $\vartheta_i^{\ k}$ in Einstein's expression, is small of the second order. The identity of $h_i^{\ kl}$ and $\mathfrak{U}_i^{\ kl}$ in harmonic coordinates generally holds only in the weak-field approximation. In the case of a spherically symmetric system we found, for instance, in section 2, eq. (2.44), that $h_i^{\ kl}$ is exactly equal to $\mathfrak{U}_i^{\ kl}$ in a system of *isotropic* coordinates, which is identical with the harmonic system only in the weak-field approximation.

A direct calculation of the "gravitational" complexes $t_i^{\ k}$ and $\vartheta_i^{\ k}$ given by (2.36) and (2.5) in harmonic coordinates also shows that they are different in general. Since $t_i^{\ k}$ and $\vartheta_i^{\ k}$ are homogeneous quadratic functions of the first-order derivatives of the field variables, we obtain consistent second-order expressions for these quantities by using the first-order equations (4.38) and (4.1), (4.24). From (3.15) and (4.38) we get to the first order

$$\Delta^i{}_{kl} = \eta^{ai}\,\frac{1}{2}\,y_{ak,l} = \frac{1}{2}\,y^i_{k,l} \quad (4.52)$$

and thus, from (3.14), (3.12), (4.48), to the second order,

$$t_i^{\ k} = \frac{1}{2\varkappa}\left\{\frac{1}{2}\,(y_l^{m,k} - y_l^{k,m})\,y^l_{m,i} + \frac{1}{4}\,(y_{,l}\,y^{lk}{}_{,i} - y_{,i}\,y^{,k}) - \delta^k_i\,\hat{\mathfrak{L}}\right\} \quad (4.53)$$

with

$$\hat{\mathfrak{L}} = \frac{1}{4}\,y_{rs,t}\,(y^{rs,t} - y^{ts,r}) - \frac{1}{16}\,y_{,r}\,y^{,r}. \quad (4.54)$$

The corresponding expression (C.23) for $\vartheta_i^{\ k}$, calculated in Appendic C, is seen to differ from $t_i^{\ k}$ as given by (4.53) the difference being given by (C.28–29). If we have consistent asymptotic expressions for the metric tensor in harmonic coordinates for an insular system of matter in vibrational motion, we can calculate the loss of energy $-\frac{dE}{dt}$ per unit time by means of the formula

$$-\frac{dE}{dt} = -c\iint\limits_{\sigma} t_4^{\ \varkappa}\,d\sigma_\varkappa, \quad (4.55)$$

where $t_4^{\ k}$ is given by (4.53) and

$$d\sigma_\varkappa = \delta_{\varkappa\lambda\mu}\,(dx^\lambda\,\delta x^\mu - dx^\mu\,\delta x_\lambda) \quad (4.56)$$

represents a surface element of a large sphere σ enclosing the system. The difficulty is only to obtain consistent asymptotic expressions for the metric tensor of a system of vibrating matter. In particular it is clear that the energy loss is given correctly by the simple expressions (4.53–56) only if the metric tensor used satisfies the field equations (2.7) up to at least the second-order terms. This again means that we have to take into account the influence of the first-order field on the motion of the vibrating matter.

The result of the investigations of this section is that the relative orientation of the tetrads is uniquely determined by the supplementary equations (4.37) together with the boundary conditions A and B. It is true that this has been shown only in the weak-field approximation, where the differential equations (4.37) become linear. However, it is a common experience that non-linear equations are more restrictive than the first-order linear approximative equations, so that the uniqueness of the solution may be assumed to hold also for the exact equations (4.37). One might rather fear that the exact equations have no regular solutions at all, but we have at least one case for which we know the exact solution, viz., the spherically symmetric case. In isotropic coordinates the tetrads $\underset{a}{h}{}^{i}$ are here given by (2.43). As shown in Appendix B, these functions $\underset{a}{h}{}^{i} = \frac{1}{\sqrt{|g_{aa}|}}\,\delta^i_a$ satisfy the equations (4.37) exactly, and, since asymptotically

$$g_{aa} = \{1+\alpha/r,\ 1+\alpha/r,\ 1+\alpha/r,\ -(1-\alpha/r)\},$$

the functions $\underset{a}{h}{}^{i}$ also satisfy the boundary conditions A, B.

It is therefore reasonable to assume that, for any insular system, the supplementary conditions (4.37) together with the boundary conditions A, B will furnish a unique determination of the relative orientation of the tetrads, which was left open by the field equations (2.31). Then also the expression for the energy-momentum complex is unique, and the problem of the energy distribution in gravitational fields can be regarded as solved.

5. Absolute Parallelism

In the usual formulation of the general theory of relativity the space-time continuum is a Riemannian space with a metric determined by the distribution of matter through Einstein's field equations, and the gravitational field is described by the metric tensor $g_{ik}(x)$. In a space of this type the parallel displacement of a vector A^i along a given curve is determined by the definition of Levi-Civita

$$d_p A^i = -\Gamma^i_{kl} A^k dx^l, \tag{5.1}$$

where Γ^i_{kl} is the Christoffel symbol. As is well known, the result of a parallel displacement along a curve connecting two distant points in space will in general depend on the form of the connecting curve. Consequently, it has no unambiguous meaning to speak of parallelism of two vectors in distant points in such a space.

In the formulation of the theory given in the preceding sections the situation is quite different in this respect. Here, the basic variables are the tetrad field variables $\underset{a}{h}^i(x)$, which define the metric by (2.19). Now, as usual, the metric is determined by Einstein's field equations (2.7) or (2.31) which, however, determine the tetrads only up to an arbitrary independent rotation of the tetrads in the different space-time points. Therefore, the equations (2.31) had to be supplemented by a set of conditions, which we have taken to be the equations (4.37) together with the boundary conditions *A*, *B* of section 3. They determine the relative orientation of the tetrads, which means that the tetrad-lattice is uniquely determined apart from an arbitrary *constant* rotation of the tetrads in the lattice. In this formulation, the space-time continuum is not an ordinary Riemannian space, but rather a space of the type first investigated by WEITZENBÖCK(5), which has a metric of the Riemannian type, but in which the notion of direction has a meaning for the space as a whole. In fact, in a space with an underlying tetrad field, two vectors at distant points may be called parallel if they have identical components with respect to the local tetrads at the points considered.

Now, the tetrad components of a vector A^i are the invariants

$$\overset{a}{A}(x) = \overset{a}{h}_k(x) A^k(x). \tag{5.2}$$

Therefore, parallel displacement in this absolute sense is defined by

$$d_{ap.} \overset{a}{A} = \overset{a}{h}_k d_{ap.} A^k + \overset{a}{h}_{k,\,l} A^k dx^l = 0 \tag{5.3}$$

or by

$$d_{ap.} A^i = -\underset{a}{h}^i \overset{a}{h}_{k,\,l} A^k dx^l = -\Delta^i{}_{kl} A^k dx^l, \tag{5.4}$$

where $\Delta^i{}_{kl}$ is given by (3.15). It is seen that the Christoffel symbols Γ^i_{kl} and the symbol Δ^i_{kl} play similar parts with respect to the two different kinds of parallel displacement defined by (5.1) and (5.4), respectively. Along with the usual covariant derivative (of the first kind) of a vector field (denoted by a semicolon) we can therefore here define a covariant derivative of the second kind by

$$\left.\begin{aligned} A^i{}_{|k} &= A^i{}_{,k} + A^r \Delta^i{}_{rk} \\ A_{i|k} &= A_{i,k} - A_r \Delta^r{}_{ik}, \end{aligned}\right\} \tag{5.5}$$

with obvious generalizations for tensors of higher rank. The usual rules about covariant derivatives also hold for the derivatives of the second kind, in particular the rule concerning differentiation of a product of two tensors and the relation

$$g_{ik|l} \equiv 0. \tag{5.6}$$

This identity follows by (2.19) from the equations

$$\underset{a}{h}{}^{i}{}_{|k} \equiv 0, \quad \underset{a}{h}{}_{i|k} \equiv 0 \tag{5.7}$$

which are immediate consequences of the definitions (5.3-4) or of (5.5), (3.15).

From the definition of the covariant derivative of the first kind, viz.,

$$\underset{a}{h}{}_{k;l} = \underset{a}{h}{}_{k,l} - \underset{a}{h}{}_{r}\Gamma^{r}_{kl}, \tag{5.8}$$

we obtain by multiplication by $\overset{a}{h}{}^{i}$ and summation over a, on account of (2.18), (3.7) and (3.15),

$$\Delta^{i}{}_{kl} = \Gamma^{i}_{kl} + \gamma^{i}{}_{kl}. \tag{5.9}$$

This gives the connection between the two kinds of covariant derivatives. For a vector field we get

$$\left.\begin{aligned} A^{i}{}_{|k} &= A^{i}{}_{;k} + A^{r}\gamma^{i}{}_{rk} \\ A_{i|k} &= A_{i;k} - A_{r}\gamma^{r}{}_{ik}, \end{aligned}\right\} \tag{5.10}$$

with obvious generalizations for tensors of higher order.

By means of these relations it is easily seen that the antisymmetric tensor ξ_{ik} entering in the supplementary equations (4.37) is equal to the divergence of the second kind of the tensor γ_{ikl}, i. e.,

$$\xi_{ik} = \gamma_{ik}{}^{l}{}_{|l} \tag{5.11}$$

(see (D.3) in Appendix D). Further, the tensors occurring on the left-hand side of Einstein's field equations may be written

$$\left.\begin{aligned} R_{ik} &= \gamma^{r}{}_{ik|r} - \Phi_{i|k} - \Phi_{r}\gamma^{r}{}_{ik} + \gamma^{r}{}_{is}\gamma^{s}{}_{kr} \\ R = R^{i}_{i} &= -2\Phi^{r}{}_{|r} + \Phi_{r}\Phi^{r} + \gamma_{rst}\gamma^{tsr} \end{aligned}\right\} \tag{5.12}$$

which, on account of the symmetry of the contracted curvature tensor, leads to the identity

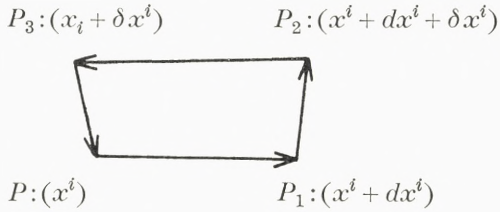
$$\gamma^{r}{}_{ik|r} - \gamma^{r}{}_{ki|r} - \Phi_{i,k} + \Phi_{k,i} = 0. \tag{5.13}$$

Similarly, the curvature tensor R_{iklm} may be expressed in terms of the tensor γ_{ikl} and its covariant derivatives of the second kind (see (D.6–8) in Appendix D).

In a space of the Weitzenböck type the most important geometrical notion is the "torsion". Consider an arbitrary closed curve; divide it into an infinite number of infinitesimal line elements. Each of these line elements is an infinitesimal vector. Perform an absolute parallel displacement of all these vectors to a given point P on the curve. The sum of all these vectors will then be a vector, the torsion vector $t^{i}(P)$, which in general is different from zero. In particular, let the closed curve be an infinitesimal "parallelogram" with corners at the points

$P:(x^i)$; $P_1:(x^i+dx^i)$;

$P_2:(x^i+dx^i+\delta x^i)$; $P_3:(x^i+\delta x^i)$



By means of (5.4) we now perform a parallel displacement of the vectors $\overrightarrow{P_1 P_2}:(\delta x^i)$, $P_2 P_3:(-dx^i)$ and $P_3 P:(-\delta x^i)$ to the point P and get for the infinitesimal torsion vector dt^i in P corresponding to the curve, neglecting terms of the second order in dx^i and δx^i,

$$dt^i = -\Delta^i{}_{kl}(dx^k \delta x^l - \delta x^k dx^l) = -\frac{1}{2} \underset{a}{h}{}^i \left(\overset{a}{h}_{k,l} - \overset{a}{h}_{l,k}\right) d\sigma^{kl} = -\frac{1}{2}(\gamma^i{}_{kl} - \gamma^i{}_{lk})\, d\sigma^{kl}$$

or

$$dt^i = -\gamma^i_{kl}\, d\sigma^{kl}, \tag{5.14}$$

where

$$d\sigma^{kl} = dx^k \delta x^l - dx^l \delta x^k \tag{5.15}$$

is the tensor representing the area of the parallelogram.

If the torsion is zero for any closed curve, we must have

$$\gamma^i_{kl} = 0 \tag{5.16}$$

everywhere. Then, also $R_{iklm} = 0$ and by (3.11)

$$\underset{a}{h}_{i;k} = 0, \tag{5.17}$$

Thus, in this case, space-time is flat and the tetrad field forms a pseudo-Cartesian lattice. This corresponds to a completely empty space. On the other hand, in the empty space surrounding an insular matter system, R_{ik} is equal to zero, but the torsion, i. e., $\gamma^i{}_{kl}$ $\underset{a}{h}_{i;k}$ and T_i^k will in general be different from zero.

6. On a Possible Generalization of the Theory

The considerations of the preceding section show that the present theory has some traits, in particular the idea of "Fernparallelismus", in common with Einstein's attempts at a unified theory of gravitation and electromagnetism of thirty years ago[(6)]. In both theories the sixteen functions $\underset{a}{h}{}^i(x)$ are taken as the fundamental field variables instead of the ten variables $g_{ik}(x)$. However, in the present paper, the object of this step was simply to arrive at a consistent expression for the energy-momentum complex, and the surplus degrees of freedom were therefore removed again by the six equations (4.37) which were regarded as supplementary field equations. Although this procedure, which consists in enlarging the number of degrees of freedom of a system and afterwards restricting it again by means of subsidiary conditions, is used also in other fields of physics (for instance in quantum electrodynamics), it would undoubtedly be more satisfactory if the tetrad field could be thought of as describing a larger domain of physics than mere gravitation. Obviously the number of the tetrad variables is just sufficient to describe the unified field of gravitation and electromagnetism, and the problem arises how to find the field equations of this unified field. Since it is desirable to maintain the usual general relativity description of gravitational and electromagnetic phenomena, we shall take over the field equations (2.31)

$$R_{ik}-\frac{1}{2}g_{ik}R=-\varkappa(T_{ik}^{n.e.m.}+S_{ik}), \tag{6.1}$$

the left-hand side of which is given for instance by (5.12), while the matter tensor on the right-hand side consists of a non-electromagnetic part and of the electromagnetic energy-momentum tensor

$$S_{ik}=F_{il}F_k^{\ l}-\frac{1}{4}g_{ik}F_{lm}F^{lm}. \tag{6.2}$$

Here, F_{ik} is the electromagnetic field tensor satisfying Maxwell's equations

$$\frac{1}{\sqrt{-g}}(\sqrt{-g}F^{ik})_{,k}\equiv F^{ik}{}_{;k}=s^i \tag{6.3}$$

$$F_{ik,l}+F_{kl,i}+F_{li,k}=0. \tag{6.4}$$

The left-hand sides of these equations satisfy two identities so that only six of the eight equations (6.3-4) are really independent. This is just sufficient to determine the six functions $F_{ik}(x)$ for given initial or boundary conditions.

Now, since the quantity ξ_{ik} on the left-hand side of (4.37) is an antisymmetrical tensor, the assumption suggests itself that the electromagnetic field tensor is proportional to ξ_{ik}. From this point of view, the supplementary equations (4.37) of sections 2-5 are then not field equations, but express a definite physical situation, viz., the case with no electromagnetic fields present. In general they would have to be replaced by the equations

$$F_{ik}(x)=\alpha\,\xi_{ik}(x), \tag{6.5}$$

where α is a universal constant of the dimensions of an electric charge. This is most easily seen in a system of coordinates in which the (x^i) have the dimensions of a length l, for then g_{ik} and $\underset{a}{h}{}^i$ are dimensionless quantities, and by (3.7) and (3.18) the quantities in (6.5) have the dimensions

$$F_{ik}\sim \text{charge}/l^2,\ \ \xi_{ik}\sim 1/l^2. \tag{6.6}$$

The insertion of (6.5) into (6.3-4) and (6.1-2) gives a set of sixteen equations which, together with suitable boundary conditions, may be supposed to determine the tetrad field uniquely (apart from a constant rotation of the tetrads that has no physical meaning). As usual, the four Bianchi identities, which are satisfied by the left-hand side of (6.1), take care of the arbitrariness in the solutions $\underset{a}{h}^i(x)$ corresponding to the arbitrariness in the choice of the system of coordinates.

The field equations (6.1-5) are quite different from the equations discussed by EINSTEIN in reference [6], which, incidentally, were shown by EINSTEIN and MAYER[11] to be incompatible with the Schwarzschild solution in the static, spherically symmetric case. As mentioned in the Introduction, they are essentially identical with a set of equations given by T. LEVI-CIVITA[7], which, however, never received proper attention. The reason for this was probably that the unification of gravitation and electromagnetism obtained in Levi-Civita's paper appeared to be purely formal. In particular, the constant α could be given no immediate physical meaning since no physical effects could be mentioned that would depend on the value of α. In this respect, the situation seems to be different in the present formulation, since the value of α will enter in the expression for the energy-momentum complex $\mathsf{T}_i^{\ k}$ and therefore will have an influence on the energy distribution. In so far as the latter can be regarded as a measurable quantity, α will then also appear as a real physical constant. However, further investigations are needed before we can see whether a generalization of the formalism of sections *2*-*5* along the lines indicated in section *6* can be carried through in a consistent way.

Finally it should be mentioned that, already in 1959[12], B. LAURENT introduced a superpotential Λ_i^{kl} which is a true tensor density. In fact, Λ_i^{kl} is equal to the first term of our expression for $\mathfrak{U}_i^{kl}$, i. e.,

$$\Lambda_i^{kl} = |h|\,\gamma^{kl}{}_i.$$

It is easily seen that this superpotential leads to a complex satisfying the conditions *1'*, *2* and *4* of section *2*. However, for a closed system it does not have the asymptotic behaviour (2.48), and consequently, like our old expression $\mathcal{T}_i^{\ k}$, it will *not* satisfy the essential condition *3*.

I want to thank Professor M. MAGNUSSON and Dr. C. PELLEGRINI, who took part in our discussions at the initial stage of this investigation and have checked many of my calculations. My thanks are further due Professor POUL KRISTENSEN for friendly and illuminating discussions, in particular on the question of the boundary conditions.

Note added in proof:

Since the completion of this paper we have made a more detailed investigation into the consistency of the generalization of the theory outlined in the last section.

An application of the equations (6.1-5) of section *6* to the simple case of a static, spherically distributed electric charge now shows that the connection between the electromagnetic field and the tetrad field, contained in the simple relation (6.5) of Levi-Civita, is inconsistent with the basic requirement 3. of section 2. The reason for this is simply the following. In order that condition 3. be satisfied, the first-order derivations of the tetrad functions in isotropic coordinates have to vanish at least as $1/r^2$ for $r \to \infty$. Then, as is easily understood, ξ_{ik} will vanish as $1/r^3$ at spatial infinity. However, the solution of the static, spherically symmetric case given by Weyl and Nordström shows that the electric field vector, and hence F_{ik}, vanishes only as $1/r^2$ for $r \to \infty$, which means that the equation (6.5) is incompatible with condition 3. On the other hand, a solution of the equation (6.5) leads to a tetrad field with first-order derivations vanishing slowlier than $1/r^2$. As a consequence, the corresponding $\mathfrak{U}_4^{\ 4\lambda}$ will contain a term of order $1/r$ at spatial infinity, which leads to a diverging value for the total energy $E = -\int \mathfrak{T}_4^{\ 4}\, dx^1\, dx^2\, dx^3$.

This result does not necessarily mean that we have to abandon the idea of a connection between the tetrad field and the electromagnetic field; it only means that this connection cannot be given by the simple relation (6.5). If we only require general covariance, there are obviously many ways of expressing such a connection. We might, for instance, substitute equation (6.5) by the relations

$$\alpha \xi_{ik} = R_{iklm} F^{lm} \tag{6.6}$$

or, even simpler, by

$$\alpha \xi_{ik} = R F_{ik}. \tag{6.7}$$

Neither of these equations would lead to the above mentioned difficulty, since both R_{iklm} and R vanish, at least as $1/r^3$, so that the right-hand sides of the equations vanish sufficiently rapidly for $r \to \infty$. In the case of the latter equation (6.7), the relation (4.37), adopted in sections 2-5, would even hold everywhere in the empty space surrounding the matter, but these equations would hold throughout the whole space only in the case when there are no electromagnetic fields present. The equations (6.6-7) have the simplicity that the right-hand sides are completely determined by Einstein's field equations and Maxwell's equations, and they are invariant under the group of Lorentz rotations (2.22-24), but one could also imagine more general connections between the electromagnetic field and the tetrad field. Perhaps the requirement that all field equations should be derivable from a variational principle may be used as a guide in our choice between the various possibilities.

Appendix A

In order to obtain the expressions for the different curvature tensors in terms of the tetrad fields we proceed in the following way. We apply the general commutation rule for covariant differentiation, i. e.,

$$A_{k;l;m} - A_{k;m;l} = -A_r R^r{}_{klm} \tag{A.1}$$

holding for any vector field A_k, to the vector field $\underset{a}{h}_k$. After multiplication of the resulting equation by $\overset{a}{h}{}^i$ we then get, using (2.18),

$$R^i{}_{klm} = \overset{a}{h}{}^i\left(\underset{a}{h}_{k;m;l} - \underset{a}{h}_{k;l;m}\right) = \overset{a}{h}_k\left(\underset{a}{h}{}^i{}_{;l;m} - \underset{a}{h}{}^i{}_{;m;l}\right). \tag{A.2}$$

Hence, for the contracted curvature tensor

$$R_{ik} = R^r{}_{irk} = \overset{a}{h}{}^r\left(\underset{a}{h}_{i;k;r} - \underset{a}{h}_{i;r;k}\right) = \overset{a}{h}_i\left(\underset{a}{h}{}^r{}_{;r;k} - \underset{a}{h}{}^r{}_{;k;r}\right) \tag{A.3}$$

and the curvature scalar

$$R = R^s_s = \overset{a}{h}{}^r\left(\underset{a}{h}{}^s{}_{;s;r} - \underset{a}{h}{}^s{}_{;r;s}\right) = \left(\overset{a}{h}{}^r\underset{a}{h}{}^s{}_{;s}\right)_{;r} - \left(\overset{a}{h}{}^r\underset{a}{h}{}^s{}_{;r}\right)_{;s} - \overset{a}{h}{}^r{}_{;r}\underset{a}{h}{}^s{}_{;s} + \overset{a}{h}{}^r{}_{;s}\underset{a}{h}{}^s{}_{;r}. \tag{A.4}$$

If we multiply this equation by $\sqrt{-g} = |h|$ we get the equations (2.26-27) in the text, i. e.,

$$\mathfrak{R} = \hat{\mathfrak{L}} + \hat{\mathfrak{h}} \tag{A.5}$$

with

$$\hat{\mathfrak{L}} = |h|\left(\overset{a}{h}{}^r{}_{;s}\underset{a}{h}{}^s{}_{;r} - \overset{a}{h}{}^r{}_{;r}\underset{a}{h}{}^s{}_{;s}\right), \tag{A.6}$$

$$\left.\begin{aligned}\hat{\mathfrak{h}} &= \left[\sqrt{-g}\left(\overset{a}{h}{}^r\underset{a}{h}{}^s{}_{;s} - \overset{a}{h}{}^s\underset{a}{h}{}^r{}_{;s}\right)\right]_{,r}\\ &= 2\left[\sqrt{-g}\,\overset{a}{h}{}^r\underset{a}{h}{}^s{}_{;s}\right]_{,r} = 2\left[\overset{a}{h}{}^r\left(|h|\,\underset{a}{h}{}^s\right)_{,s}\right]_{,r}.\end{aligned}\right\} \tag{A.7}$$

Here we have used the relation

$$\underset{a}{h}{}^r{}_{;s}\overset{a}{h}{}^s = -\underset{a}{h}{}^r\overset{a}{h}{}^s{}_{;s} = -\overset{a}{h}{}^r\underset{a}{h}{}^s{}_{;s} \tag{A.8}$$

following from (2.19) and the identity $g^{rs}{}_{;s} \equiv 0$.

From the definitions of the covariant derivatives and the Christoffel symbols we get, by means of (2.15-19),

$$\left.\begin{aligned}\underset{a}{h}_{k;l} &= \underset{a}{h}_{k,l} - \underset{a}{h}_i\Gamma^i_{kl} = \underset{a}{h}_{k,l} - \frac{1}{2}\underset{a}{h}{}^i\left[\left(\underset{b}{h}_i\overset{b}{h}_k\right)_{,l} + \left(\underset{b}{h}_i\overset{b}{h}_l\right)_{,k} - \left(\underset{b}{h}_k\overset{b}{h}_l\right)_{,i}\right]\\ &= \frac{1}{2}\left\{\left(\underset{a}{h}_{k,l} - \underset{a}{h}_{l,k}\right) - \underset{a}{h}{}^i\left[\overset{b}{h}_l\left(\underset{b}{h}_{i,k} - \underset{b}{h}_{k,i}\right) + \overset{b}{h}_k\left(\underset{b}{h}_{i,l} - \underset{b}{h}_{l,i}\right)\right]\right\}\end{aligned}\right\} \tag{A.9}$$

from which the tensor character of $\underset{a}{h}_{k;l}$ is apparent, in spite of the fact that $\underset{a}{h}_{k;l}$ appears as a linear function of the ordinary derivatives of the tetrads. The equation (A.9) may also be written

$$\underset{a}{h}_{k;l} = \frac{1}{2}\underset{a}{h}{}^i P_{ikl}{}^{rst}\overset{b}{h}_r \underset{b}{h}_{s,t}, \tag{A.10}$$

where $P_{ikl}{}^{rst}$ is a tensor of the form

$$P_{ikl}{}^{rst} = \delta^r_i g_{kl}{}^{st} + \delta^r_k g_{li}{}^{st} - \delta^r_l g_{ik}{}^{st} \tag{A.11}$$

and $g_{kl}{}^{st}$ is the tensor

$$g_{kl}{}^{st} = \delta^s_k \delta^t_l - \delta^s_l \delta^t_k, \tag{A.12}$$

satisfying the same symmetry relations as the curvature tensor, i. e.,

$$\left.\begin{aligned} g_{klst} = g_{ks}g_{lt} - g_{ls}g_{kt} = -g_{lkst} = -g_{klts} = g_{stkl} \\ g_{klst} + g_{kstl} + g_{ktls} = 0. \end{aligned}\right\} \tag{A.13}$$

From (2.18) we get at once by differentiation the following connection between the derivatives of the covariant and the contravariant components of the tetrad vectors:

$$\underset{b}{h}_{s,t} = -\underset{a}{h}_s \overset{a}{h}{}^u{}_{,t}\underset{b}{h}_u \tag{A.14}$$

or, by (2.19),

$$\overset{b}{h}_r \underset{b}{h}_{s,t} = -g_{ru}\underset{b}{h}_s \overset{b}{h}{}^u{}_{,t}. \tag{A.15}$$

Thus, (A.10) may be written

$$\underset{a}{h}_{k;l} = -\frac{1}{2}\underset{a}{h}{}^i P_{iklr}{}^{st}\underset{b}{h}_s \overset{b}{h}{}^r{}_{,t} \tag{A.16}$$

and, since $P_{iklr}{}^{st}$ does not depend on the derivatives of the tetrad functions, we find

$$\frac{\partial \overset{a}{h}_{k;l}}{\partial \overset{b}{h}{}^r{}_{,t}} = -\frac{1}{2}\overset{a}{h}{}^i P_{iklr}{}^{st}\underset{b}{h}_s \tag{A.17}$$

which is seen to be a tensor. This means that the indices in this expression may be raised and lowered according to the usual tensor rules. Therefore we get from (A.6) for the derivative of $\hat{\mathfrak{L}}$ with respect to $\overset{a}{h}{}^i{}_{,l}$

$$\left.\begin{aligned} \frac{\partial \hat{\mathfrak{L}}}{\partial \overset{a}{h}{}^i{}_{,l}} &= 2\,|h|\left[\underset{b}{h}{}^t{}_{;s}\frac{\partial \overset{b}{h}{}^s{}_{;t}}{\partial \overset{a}{h}{}^i{}_{,l}} - \underset{b}{h}{}^r{}_{;r}\frac{\partial \overset{b}{h}{}^s{}_{;s}}{\partial \overset{a}{h}{}^i{}_{,l}}\right] \\ &= -\,|h|\left[\underset{b}{h}{}^t{}_{;s}\overset{b}{h}{}^r P_r{}^s{}_{ti}{}^{kl}\underset{a}{h}_k - \left(\underset{b}{h}{}^s{}_{;s}\right)\overset{b}{h}{}^r P_r{}^s{}_{si}{}^{kl}\underset{a}{h}_k\right]. \end{aligned}\right\} \tag{A.18}$$

Now, from (A.11-12), we get

$$P_{r\ ti}^{\ s\ \ kl} = g_{ir}(g^{ks}\delta_t^l - \delta_t^k g^{ls}) + \delta_i^s(\delta_t^k\delta_r^l - \delta_r^k\delta_t^l) - g_{it}(\delta_r^k g^{ls} - g^{ks}\delta_r^l) \tag{A.19}$$

$$P_{r\ si}^{\ s\ \ kl} = 2(\delta_i^k\delta_r^l - \delta_i^l\delta_r^k) \tag{A.20}$$

and by introduction into (A.18) we obtain

$$\left.\begin{aligned} \frac{\partial\hat{\mathfrak{L}}}{\partial \overset{a}{h}{}^i{}_{,l}} &= -|h|\left[\overset{b}{h}{}^l \underset{b}{h}{}^k{}_{;i} - \overset{b}{h}{}^k \underset{b}{h}{}^l{}_{;i} - 2\left(\underset{b}{h}{}^s{}_{;s}\right)\left(\delta_i^k \overset{b}{h}{}^l - \delta_i^l \overset{b}{h}{}^k\right)\right]\underset{a}{h}{}_k \\ &= 2|h|\left[\overset{b}{h}{}^r \underset{b}{h}{}^l{}_{;i} + \left(\underset{b}{h}{}^s{}_{;s}\right)\left(\delta_i^r \overset{b}{h}{}^l - \delta_i^l \overset{b}{h}\right)\right]\underset{a}{h}{}_r . \end{aligned}\right\} \tag{A.21}$$

The method of infinitesimal transformations applied to the variant $V = \hat{\mathfrak{L}}/2k$ with the functions $\overset{a}{h}{}^r$ as independent variables gives at once (see C. Møller, reference 3, Eq. (33)) the expressions (2.35-36) for the complex $\mathsf{T}_i^{\ k}$. Similarly, one finds from the equations (34)-(38) in the same paper that the superpotential $\mathfrak{U}_i^{\ kl}$ is given by (2.38) and that this expression is antisymmetrical in k and l. In these derivations we have used that $\overset{a}{h}{}^r$ is a vector, which means that its variation under an infinitesimal co-ordinate transformation

$$\bar{x}^i = x^i + \xi^i(x) \tag{A.22}$$

is given by

$$\delta \overset{a}{h}{}^r = \overset{a}{h}{}^k \xi^r{}_{,k} - \overset{a}{h}{}^r{}_{,i}\xi = \delta_i^r \overset{a}{h}{}^k \xi^i{}_{,k} - \overset{a}{h}{}^r{}_{,i}\xi^i . \tag{A.23}$$

If we now introduce the expression (A.21) for the derivatives of $\hat{\mathfrak{L}}$ with respect to the first-order derivatives of the tetrads into (2.36) and (2.38), respectively, we are immediately led to the equations (2.39-40) in the text.

Appendix B

According to (3.7) and (A.10) we have generally the following expression of the tensor γ_{ikl} in terms of the first-order derivatives of the tetrad functions

$$\gamma_{ikl} = \overset{a}{h}_i \underset{a}{h}{}_{k;l} = \frac{1}{2} P_{ikl}^{\ \ \ rst} \overset{a}{h}_r \underset{a}{h}{}_{s,t} . \tag{B.1}$$

Similarly, for the vector Φ_k defined by (3.10),

$$\Phi_k = \gamma^i{}_{ki} = \frac{1}{2} P^i{}_{ki}{}^{rst} \overset{a}{h}_r \underset{a}{h}{}_{s,t} . \tag{B.2}$$

In particular for a static spherically symmetric system in isotropic coordinates, the metric and the "right" tetrads are given by (2.42) and (2.43), respectively. Thus

$$\overset{a}{h}_r \underset{a}{h}_{s,t} = \sqrt{|g_{aa}|}\,\delta_{ar}\,\varepsilon_a \left(\sqrt{|g_{aa}|}\right)_{,t} \delta_{as} = \frac{g_{rr),t}}{2}\,\delta_{rs} \tag{B.3}$$

and, by (B.1) and (A.11-12),

$$\gamma_{ikl} = \frac{1}{2}\left[g_{ll),i}\,\delta_{kl} - g_{ll),k}\,\delta_{il}\right] = \frac{g'_{ll)}}{2}\left(n_i\,\delta_{kl} - n_k\,\delta_{il}\right) \tag{B.4}$$

with

$$n_i \equiv n^i \equiv \frac{\partial r}{\partial x^i} = \left\{\frac{x^1}{r}, \frac{x^2}{r}, \frac{x^3}{r}, 0\right\} \tag{B.5}$$

and

$$A'_{(r)} \equiv \frac{dA(r)}{dr}. \tag{B.6}$$

Hence,

$$\left.\begin{aligned} \Phi_k = \gamma^i{}_{ki} &= \frac{g'_{ii}}{2\,g_{ii}}\left(n_i\,\delta_{ki} - n_k\right) \\ &= -\sum_{i \neq k}\left(ln\sqrt{|g_{ii}|}\right)' n_k, \end{aligned}\right\} \tag{B.7}$$

or, by (2.42) and (B.5),

$$\left.\begin{aligned} \Phi_k &= -\left[2\,(ln\sqrt{a})' + (ln\sqrt{b})'\right] n_k \\ &= -(ln\,a\sqrt{b})'\,n_k = -(ln\,a\sqrt{b})_{,k}. \end{aligned}\right\} \tag{B.8}$$

Similarly, we get by (3.15) and (2.43)

$$\left.\begin{aligned} \Delta^i{}_{kl} = \overset{a}{h}{}^i \underset{a}{h}_{k,l} &= \frac{\varepsilon_{a)}}{\sqrt{|g_{aa}|}}\,\delta^i_a\,\varepsilon_a\left(\sqrt{|g_{aa}|}\right)_{,l}\delta_{ak} \\ &= \left(ln\sqrt{|g_{ii}|}\right)_{,l}\delta_{ik} = \left(ln\sqrt{|g_{ii}|}\right)'\,\delta^{i)}_k\,n_l. \end{aligned}\right\} \tag{B.9}$$

Now we see at once from (B.8), (B.4) that the quantities η_{ik}, ξ_{ik}, τ_{ik} defined by (3.19), (3.16-17) are zero in the case of the tetrads (2.43), for we have

$$\left.\begin{aligned} \eta_{ik} &= \Phi_{k,i} - \Phi_{i,k} = -(ln\,a\sqrt{b})_{,k,i} + ln(a\sqrt{b})_{,i,k} = 0 \\ \zeta_{ik} &= \gamma_{ik}{}^l\,\Phi_l = -\frac{g'_{ll}}{2\,g_{ll}}\left(n_i\,\delta_{kl} - n_k\,\delta_{il}\right)(ln\,a\sqrt{b})'\,n_l \end{aligned}\right\} \tag{B.10}$$

or, since

$$n_4 = 0 \quad \text{and} \quad g_{ll} = a(r) \quad \text{for} \quad l = 1, 2, 3, \tag{B.11}$$

$$\zeta_{ik} = -(ln\sqrt{a})'\,(ln\,a\sqrt{b})'\left(n_i\,\delta_{kl}\,n_l - n_k\,\delta_{il}\,n_l\right) = 0. \tag{B.12}$$

Further,

$$\left.\begin{aligned} \tau_{ik} &= \Phi^l(\gamma_{lik}-\gamma_{lki}) \\ &= -\frac{(ln\,a\sqrt{b})'\,n_l}{2\,g_{ll}}\left[g_{kk}'(n_l\delta_{ik}-n_i\delta_{kl})-g_{ii}'(n_l\delta_{ki}-n_k\delta_{il})\right] \\ &= -(ln\,a\sqrt{b})'\frac{g_{ll}'}{2\,g_{ll}}(n_k\delta_{il}n_l-n_i\delta_{kl}n_l) \\ &= -(ln\,a\sqrt{b})'(ln\sqrt{a})'(n_k\delta_{il}n_l-n_i\delta_{kl}n_l)=0\,. \end{aligned}\right\} \quad (B.13)$$

Finally, since $\zeta_{ik}=0$, we get for the quantity ξ_{ik} defined by (3.18)

$$\xi_{ik}=\gamma_{ik}{}^{l}{}_{;l}=\frac{1}{h}(h\gamma_{ik}{}^{l})_{,l}+\Gamma^r_{il}\gamma_{kr}{}^{l}-\Gamma^r_{kl}\gamma_{ir}{}^{l} \quad (B.14)$$

and it is easily seen that this is equal to zero for the tetrads given by (2.43). First, we get by (B.4) and (2.42)

$$\left.\begin{aligned} (h\gamma_{ik}{}^{l})_{,l} &= \left[\frac{hg_{ll}'}{2\,g_{ll}}(n_i\delta_{kl}-n_k\delta_{il})\right]_{,l} \\ &= [h(ln\sqrt{a})'n_i]_{,k}-[h(ln\sqrt{a})'n_k]_{,i}=0, \end{aligned}\right\} \quad (B.15)$$

since $n_{i,k}=r,i,k$ is symmetrical in i and k.

Further, we have by (2.42)

$$\Gamma^r_{il}=\frac{1}{2\,g_{rr)}}(\delta_{ri}g_{rr}'n_l+\delta_{rl}g_{rr}'n_i-\delta_{il}g_{ii}'n_r)$$

and thus by (B.4)

$$\left.\begin{aligned} \Gamma^r_{il}\gamma_{kr}{}^{l} &= (ln\sqrt{|g_{ii}|})'n_l\gamma_{ki}{}^{l}+(ln\sqrt{|g_{ll}|})'n_i\gamma_{kl}{}^{l}-\frac{g_{ii}'}{2\,g_{rr}}n_r\gamma_{kr}{}^{i} \\ &= (ln\sqrt{|g_{ii}|})'\frac{a'}{2\,a}(n_k\delta_{il}-n_i\delta_{kl})n_l \\ &\quad+(ln\sqrt{|g_{ll}|})'n_i\frac{g_{ll}'}{2\,g_{ll}}(n_k-n_l\delta_{kl}) \\ &\quad-\frac{g_{ii}'}{2\,a}\frac{n_r g_{ii}'}{2\,g_{ii}}(n_k\delta_{ri}-n_r\delta_{ki}) \\ &= \sum_{l\neq k}[(ln\sqrt{|g_{ll}|})']^2 n_i n_k-\frac{(g_{ii}')^2}{4\,a\,g_{ii}}(n_i n_k-\delta_{ik}) \\ &= \left(2\,[ln\sqrt{a})']^2+[(ln\sqrt{b})']^2-\frac{a'^2}{4\,a^2}\right)n_i n_k+\frac{(g_{ii}')^2}{4\,a\,g_{ii}}\delta_{ik}. \end{aligned}\right\} \quad (B.16)$$

Here we have again used the fact that n_k is different from zero only for $k = 1, 3, 3$. Since the expression (B.16) is symmetrical in i and k, the last two terms in (B.14) cancel, which together with (B.15) shows that also

$$\xi_{ik} = 0 \tag{B.17}$$

for the tetrads (2.43) in the static spherically symmetric case.

We now get by (B.4) and (B.8) for the superpotential (3.13) in the static spherically symmetric case

$$\left.\begin{aligned} \mathfrak{U}_i^{\ kl} &= \frac{a^{3/2} b^{1/2}}{\varkappa}\left[\frac{g_{ii)}'}{2\, g_{kk)}\, g_{ll)}}, (\delta_i^l n^k - \delta_i^k n^l) + (ln\, a\sqrt{b})' \left(\delta_i^k \frac{n^l}{g_{ll)}} - \delta_i^l \frac{n^k}{g_{kk)}}\right)\right] \\ &= \frac{\sqrt{ab}}{\varkappa}\left(ln \frac{a\sqrt{b}}{\sqrt{|g_{ii}|}}\right)' (\delta_i^k n^l - \delta_i^l n^k), \end{aligned}\right\} \tag{B.18}$$

i. e., equation (2.44) in the text. The same expression is obtained for the superpotential $h_i^{\ kl}$ of the Einstein expression in isotropic coordinates. In fact, we have by (2.6) and (2.42)

$$\left.\begin{aligned} h_i^{\ kl} &= \frac{g_{in}}{2\varkappa\sqrt{-g}} [(-g)(g^{kn} g^{lm} - g^{ln} g^{km})]_{,m} \\ &= \frac{g_{ii)}}{2\varkappa a^{3/2} b^{1/2}} \left[\left(\frac{a^3 b}{g_{kk)}\, g_{ll)}}\right)_{,l} \delta_i^k - \left(\frac{a^3 b}{g_{kk)}\, g_{ll)}}\right)_{,k} \delta_i^l\right] \\ &= \frac{\sqrt{ab}}{\varkappa}\left(\frac{g_{ii}}{2\, a^2 b}\right)\left(\frac{a^2 b}{g_{ii)}}\right)' [\delta_i^k n^l - \delta_i^l n^k] \\ &= \frac{\sqrt{ab}}{\varkappa}\left(ln \frac{a\sqrt{b}}{\sqrt{|g_{ii}|}}\right)' (\delta_i^k n^l - \delta_i^l n^k). \end{aligned}\right\} \tag{B.19}$$

For $k = 4$, $l = \lambda$ we have by (B.18)

$$\mathfrak{U}_i^{\ 4\lambda} = \frac{\sqrt{ab}}{\varkappa}\left(ln \frac{a\sqrt{b}}{\sqrt{|g_{ii}|}}\right)' \delta_i^4 n^\lambda = \frac{\sqrt{ab}}{\varkappa} (ln\, a)' \delta_i^4 n^\lambda = -\mathfrak{U}_i^{\ \lambda 4}. \tag{B.20}$$

Similarly,

$$\left.\begin{aligned} \mathfrak{U}_i^{\ \varkappa\lambda} &= \frac{\sqrt{ab}}{\varkappa}\left(ln \frac{a\sqrt{b}}{\sqrt{|g_{ii}|}}\right)' (\delta_i^\varkappa n^\lambda - \delta_i^\lambda n^\varkappa) = \frac{\sqrt{ab}}{\varkappa} (ln\sqrt{ab})' (\delta_i^\varkappa n^\lambda - \delta_i^\lambda n^\varkappa) \\ &= \frac{\sqrt{ab}'}{\varkappa} (\delta_i^\varkappa n^\lambda - \delta_i^\lambda n^\varkappa). \end{aligned}\right\} \tag{B.21}$$

From these expressions we get, since

$$n^{\varkappa}{}_{,\lambda} = \frac{\partial}{\partial x^{\lambda}} \frac{x^{\varkappa}}{r} = \frac{\delta^{\varkappa}_{\lambda}}{r} - \frac{x^{\varkappa}}{r^2} n_{\lambda} = \frac{\delta^{\varkappa}_{\lambda} - n^{\varkappa} n_{\lambda}}{r}, \qquad n^{\lambda}{}_{,\lambda} = \frac{2}{r}, \qquad n^{\lambda} n_{\lambda} = 1, \tag{B.22}$$

$$\left.\begin{aligned} \mathsf{T}_i^{\,4} = \mathfrak{U}_i^{\,4\lambda}{}_{,\lambda} &= \frac{1}{\varkappa}\left\{[\sqrt{ab}\,(\ln a)']' + \frac{2\sqrt{ab}\,(\ln a)'}{r}\right\}\delta_i^4 \\ &= \frac{1}{\varkappa r^2}\,[r^2\sqrt{ab}\,(\ln a)']'\,\delta_i^{\,4}, \end{aligned}\right\} \tag{B.23}$$

and

$$\left.\begin{aligned} \mathsf{T}_i^{\,\varkappa} = \mathfrak{U}_i^{\,\varkappa l}{}_{,l} = \mathfrak{U}_i^{\,\varkappa\lambda}{}_{,\lambda} &= \frac{1}{\varkappa}\sqrt{ab}''\, n_{\lambda}\,(\delta_i^{\,\varkappa} n^{\lambda} - \delta_i^{\lambda} n^{\varkappa}) \\ &+ \frac{1}{\varkappa}\sqrt{ab}'\left(\delta_i^{\varkappa}\frac{2}{r} - \delta_i^{\,\lambda}\frac{\delta^{\varkappa}_{\lambda} - n^{\varkappa} n_{\lambda}}{r}\right) \\ &= \frac{1}{\varkappa}\left\{\sqrt{ab}''\,(\delta_i^{\,\varkappa} - n_i\, n^{\varkappa}) + \frac{\sqrt{ab}'}{r}\,(\delta_i^{\,\varkappa} + n_i\, n^{\varkappa})\right\}, \\ \text{i. e.,}\qquad \mathsf{T}_i^{\,\varkappa} &= \frac{1}{\varkappa r}\,(r\sqrt{ab}')'\,\delta_i^{\,\varkappa} - \frac{r}{\varkappa}\left(\frac{\sqrt{ab}'}{r}\right)' n_i\, n^{\varkappa}. \end{aligned}\right\} \tag{B.24}$$

In the empty space surrounding the spherically distributed matter, the functions $a(r)$ and $b(r)$ are

$$a = \left(1 + \frac{\alpha}{4r}\right)^4, \qquad b = \frac{(1 - \alpha/4r)^2}{(1 + \alpha/4r)^2}, \tag{B.25}$$

where α is connected with the total gravitational mass M_0 by (2.47):

$$\alpha = \frac{2kM_0}{c^2} = \frac{\varkappa c^2}{4\pi} M_0. \tag{B.26}$$

Hence,

$$\left.\begin{aligned} \sqrt{ab} &= 1 - \left(\frac{\alpha}{4r}\right)^2, \qquad \sqrt{ab}' = \frac{\alpha^2}{8r^3} \\ (\ln a)' &= \frac{4}{1 + \alpha/4r}\left(-\frac{\alpha}{4r^2}\right) = -\frac{\alpha/r^2}{1 + \alpha/4r} \end{aligned}\right\} \tag{B.27}$$

and, by (B.23-24),

$$\mathsf{T}_i^{\,4} = -\frac{\alpha}{\varkappa r^2}\left(1 - \frac{\alpha}{4r}\right)'\delta_i^4 = -\frac{\alpha^2}{4\varkappa r^4}\,\delta_i^{\,4} \tag{B.28}$$

$$\mathsf{T}_i^{\,\varkappa} = -\frac{\alpha^2}{4\varkappa r^4}\,\delta_i^{\,\varkappa} + \frac{\alpha^2}{2\varkappa r^4}\, n_i\, n^{\varkappa} = -\frac{\alpha^2}{4\varkappa r^4}\,(\delta_i^{\,\varkappa} - 2\, n_i\, n^{\varkappa}). \tag{B.29}$$

Obviously the equations (B.28-29) may be comprised in the equation

$$\mathsf{T}_i^k = -\frac{\alpha^2}{4\varkappa r^4}(\delta_i^k - 2\, n_i\, n^k) \tag{B.30}$$

which is the expression for the energy-momentum complex in the empty space surrounding a static spherically symmetric distribution of matter in an isotropic system of coordinates. While $\mathsf{T}_i^k = \Theta_i^k$ in *isotropic* coordinates, this identity does not hold in the harmonic system of coordinates x'^i obtained by the transformation

$$x'^{\iota} = x^{\iota}\left(1 + \frac{\alpha^2}{16\, r^2}\right), \qquad x'^4 = x^4. \tag{B.31}$$

For harmonic coordinates the line element is

$$\left.\begin{aligned} ds^2 &= \left(1 + \frac{\alpha}{2\, r'}\right)^2 \sum_{\iota} (dx'^{\iota})^2 + \frac{1 + \alpha/2\, r'}{1 - \alpha/2\, r'} \cdot \frac{\alpha^2}{4\, r'^2}(n'_{\iota}\, dx'^{\iota})^2 - \frac{1 - \alpha/2\, r'}{1 + \alpha/2\, r'}\, c^2\, dt'^2 \\ n'_{\iota} &= \frac{x'^{\iota}}{r^1}, \qquad r' = \sqrt{\sum_{\iota} (x'^{\iota})^2} \end{aligned}\right\} \tag{B.32}$$

and a simple calculation shows that the tetrads (2.43) in this system have the components

$$\left.\begin{aligned} \overset{\alpha}{h}{}'_i &= \left(1 + \frac{\alpha}{2\, r'}\right)\left[\delta_i^{\alpha} + \frac{\alpha^2}{4\, r'^2}\, \frac{n'^{\alpha}\, n'_i}{\sqrt{1 - \alpha^2/4\, r'^2}\,(1 + \sqrt{1 - \alpha^2/4\, r'^2})}\right] \\ \overset{4}{h}{}'_i &= \sqrt{\frac{1 - \alpha/2\, r'}{1 + \alpha/2\, r'}}\, \delta_i^4. \end{aligned}\right\} \tag{B.33}$$

Since the transformation (B.31) is of the type (2.3), the energy density

$$h = -\mathsf{T}_4^4/\sqrt{\gamma} = \frac{\alpha^2}{4\varkappa r^4 (1 + \alpha/4\, r)^6} \tag{B.34}$$

is invariant under this transformation, while $\Theta_4^4/\sqrt{\gamma} = h_4^{4\lambda}{}_{,\lambda}/\sqrt{\gamma}$ is not. However, in the limiting case of weak fields, i. e. to the first order in α/r, the isotropic system and the harmonic systems of coordinates are seen to coincide, the line element being in both systems of the form

$$ds^2 = (1 + \alpha/r) \sum_i (dx^{\iota})^2 - (1 - \alpha/r)\, c^2\, dt^2. \tag{B.35}$$

Therefore, in the weak-field approximation, we have $\mathsf{T}_i^k = \Theta_i^k$ also in the harmonic system of coordinates.

Appendix C

In an arbitrary system of coordinates of the type (4.1), the equation (4.37) reads, according to (4.20),

$$\Box\, \nu_{ik} = -\frac{1}{2}\left(y^l_{k,\,l,\,i} - y^l_{i,\,l,\,k}\right), \tag{C.1}$$

which together with the boundary conditions B leads to the expression

$$\nu_{ik} = \frac{1}{8\pi}\int \frac{\delta\,(x^4 - x'^4 - R)}{R}\left[\, y^l_{k,\,l,\,i}(x') - y^l_{i,\,l,\,k}(x')\right] dx'. \tag{C.2}$$

Instead of (4.38) we then get, by (4.15) and (4.18), for the "right" tetrads

$$\underset{a}{h}_i = \eta_{ai} + \frac{1}{2}\, y_{ai} + \nu_{ai}, \tag{C.3}$$

where ν_{ai} is given by (C.2), i. e. in general $\underset{a}{h}_i$ is a non-local function of the metric quantities y_{ik}. Now, let $\xi^i(x)$ be the infinitesimal transformation function connecting our system of coordinates x^i with a harmonic system by

$$x^i_{\text{harm.}} = x^i + \xi^i(x). \tag{C.4}$$

Then we have for the metric tensor

$$g^{ik}_{\text{harm.}} = \eta^{ik} - y^{ik}_{\text{harm.}} = \frac{\partial x^i_{\text{harm.}}}{\partial x^l}\,\frac{\partial x^k_{\text{harm.}}}{\partial x^m}\, g^{lm} = (\delta^l_i + \xi^i{}_{,\,l})\,(\delta^k_m + \xi^k{}_{,\,m})\,(\eta^{lm} - y^{lm})$$
$$= \eta^{ik} - y^{ik} + \xi^i{}_{,\,l}\,\eta^{lk} + \xi^k{}_{,\,m}\,\eta^{im},$$

i. e.,

$$\left.\begin{aligned} y^{ik}_{\text{harm.}} &= y^{ik} - \xi^{i,\,k} - \xi^{k,\,i}, & y^{\text{harm.}}_{ik} &= y_{ik} - \xi_{i,\,k} - \xi_{k,\,i} \\ (y^k_i)_{\text{harm.}} &= y^k_i - \xi_i{}^{,\,k} - \xi^k{}_{,\,i}, & y_{\text{harm.}} &= y - 2\,\xi^l{}_{,\,l}. \end{aligned}\right\} \tag{C.5}$$

In harmonic coordinates the deDonder condition (4.24) holds, which by (C.5) leads to the following differential equations for the functions $\xi^i(x)$:

$$0 \equiv \left(y^k_{i,\,k} - \frac{1}{2}\, y_{,\,i}\right)_{\text{harm.}} = y^k_{i,\,k} - \frac{1}{2}\, y_{,\,i} - \xi_i{}^{,\,k}{}_{,\,k} - \xi^k{}_{,\,k,\,i} + y^l{}_{,\,l,\,i}$$

or

$$\Box\, \xi_i(x) = y^l_{i,\,l} - \frac{1}{2}\, y_{,\,i} \tag{C.6}$$

with the retarded solution

$$\xi_i(x) = -\frac{1}{4\pi}\int \frac{\delta\,(x^4 - x'^4 - R)}{R}\left(y^l_{i,\,l}(x') - \frac{1}{2}\, y_{,\,i}(x')\right) dx'. \tag{C.7}$$

Since the retarded Green's function $\frac{1}{R}\delta(x^4 - x'^4 - R)$ is symmetrical in (x^i) and (x'^i), we get by partial integration

$$\xi_{i,k} - \xi_{k,i} = -\frac{1}{4\pi}\int \frac{\delta(x^4 - x'^4 - R)}{R}(y^l_{i,l,k} - y^l_{k,l,i})'\, dx' = 2\nu_{ik}. \tag{C.8}$$

Hence, by (C.3),

$$\underset{a}{h_i} = \eta_{ai} + \frac{1}{2} y_{ai} + \frac{1}{2}(\xi_{a,i} - \xi_{i,a}), \tag{C.9}$$

i. e. the equations (4.40-41) in the text. The same expression is of course obtained by a simple transformation of the vector $\underset{a}{h_i}$ by means of (C.4) and by using (C.5) and the equations (4.38) valid in a harmonic system of coordinates. In fact, we get

$$\begin{aligned}\underset{a}{h_i} &= \frac{\partial x^k_{\text{harm.}}}{\partial x^i}\underset{a}{h_k^{\text{harm.}}} = (\delta^k_i + \xi^k{}_{,i})\left(\eta_{ak} + \frac{1}{2} y^{\text{harm.}}_{ak}\right) \\ &= \eta_{ai} + \xi_{a,i} + \frac{1}{2} y^{\text{harm.}}_{ai} = \eta_{ai} + \xi_{a,i} + \frac{1}{2}(y_{ai} - \xi_{a,i} - \xi_{i,a}) \\ &= \eta_{ai} + \frac{1}{2}(y_{ai} + \xi_{a,i} - \xi_{i,a}).\end{aligned}$$

For the superpotential h_i^{kl} of the Einstein complex Θ_i^k we get to the first order, by (2.6), (4.1-2) and the equation

$$g = -1 - y_{11} - y_{22} - y_{33} + y_{44} = -(1 + y), \tag{C.10}$$

$$\left.\begin{aligned}h_i^{kl} &= \frac{\eta_{in}}{2\varkappa}\left[(1+y)\left((\eta^{kn} - y^{kn})(\eta^{lm} - y^{lm}) - (\eta^{km} - y^{km})(\eta^{ln} - y^{ln})\right)\right]_{,m} \\ &= \frac{1}{2\varkappa}\left[y(\delta^k_i \eta^{lm} - \delta^l_i \eta^{km}) - y^k_i \eta^{lm} - \delta^k_i y^{lm} + \eta^{km} y^l_i + \delta^l_i y^{km}\right]_{,m} \\ h_i^{kl} &= \frac{1}{2\varkappa}\left[\delta^k_i(y^{,l} - y^{lm}{}_{,m}) - \delta^l_i(y^{,k} - y^{km}{}_{,m}) + y_i^{l,k} - y_i^{k,l}\right].\end{aligned}\right\} \tag{C.11}$$

Thus, in a harmonic system of coordinates where the y^k_i satisfy the conditions (4.24), we get

$$h_i^{kl} = \frac{1}{2\varkappa}\left[y_i^{l,k} - y_i^{k,l} + \frac{1}{2}\delta^k_i y^{,l} - \frac{1}{2}\delta^l_i y^{,k}\right], \tag{C.12}$$

which is seen to be identical with the expression (4.49) for the superpotential $\mathfrak{U}_i^{kl}$. Therefore, to the first order, we have $\Theta_i^k = \mathrm{T}_i^k$. In fact, as shown in section 4 eq. (4.51), they are both equal to the matter tensor as it should be since t_i^k in (2.35-36) is small of the second order.

By the last argument it follows that, to the first order, $\Theta_i^{\ k}$ must be equal to $\mathsf{T}_i^{\ k}$ in any system of coordinates of the type (4.1), in spite of the fact that the equality $h_i^{\ kl} = \mathfrak{U}_i^{\ kl}$ holds only in a harmonic system of coordinates. In fact, we have in a general system of coordinates, by (4.17),

$$\gamma_{ikl} = \nu_{ik,\,l} + \frac{1}{2}(y_{kl,\,i} - y_{il,\,k}), \qquad (C.13)$$

where ν_{ik} is given by (C.2) or (C.8). Hence

$$\Phi_k = \gamma^i_{\ ki} = \nu^i_{\ k,\,i} + \frac{1}{2}(y^i_{k,\,i} - y_{,\,k}) \qquad (C.14)$$

and by (3.13)

$$\left.\begin{aligned} \mathfrak{U}_i^{\ kl} = \frac{1}{2\varkappa}\{ & y_i^{l,\,k} - y_i^{k,\,l} + \delta_i^k(y^{,\,l} - y^{lm}_{\ \ ,\,m}) - \delta_i^l(y^{,\,k} - y^{km}_{\ \ ,\,m}) \\ & + 2\,\nu^{kl}_{\ \ ,\,i} - \delta_i^k\, 2\,\nu^{ml}_{\ \ ,\,m} + \delta_i^l\, 2\,\nu^{mk}_{\ \ ,\,m}\}. \end{aligned}\right\} \qquad (C.15)$$

Now, in a general system of coordinates, $h_i^{\ kl}$ is given by (C.11), so that by (C.8)

$$\mathfrak{U}_i^{\ kl} = h_i^{\ kl} + \frac{1}{2\varkappa}[\xi^{k,\,l}_{\ \ \ ,\,i} - \xi^{l,\,k}_{\ \ \ ,\,i} - \delta_i^k(\xi^{m,\,l}_{\ \ \ \ ,\,m} - \xi^{l,\,m}_{\ \ \ ,\,m}) + \delta_i^l(\xi^{m,\,k}_{\ \ \ \ ,\,m} - \xi^{k,\,m}_{\ \ \ ,\,m})] \neq h_i^{\ kl}. \qquad (C.16)$$

Nevertheless, we have to the first order

$$\mathsf{T}_i^{\ k} = \mathfrak{U}_i^{\ kl}{}_{,\,l} = h_i^{\ kl}{}_{,\,l} + \frac{1}{2\varkappa}[\xi^{k,\,l}_{\ \ \ ,\,i,\,l} - \xi^{l,\,k}_{\ \ \ ,\,i,\,l} + \xi^{m,\,k}_{\ \ \ \ ,\,m,\,i} - \xi^{k,\,m}_{\ \ \ ,\,m,\,i}] = \Theta_i^{\ k}, \qquad (C.17)$$

as we should according to the above argument.

However, already in the second order approximation, $\Theta_i^{\ k}$ is in general different from $\mathsf{T}_i^{\ k}$. In section 4 we have calculated the "gravitational" complex $t_i^{\ k}$ to the second order. In harmonic coordinates it is given by (4.53-54). We shall now calculate $\vartheta_i^{\ k}$ to the second order. The exact expression for $\vartheta_i^{\ k}$ is (see, for instance, reference 13, Chapter XI, Eqs. (158), (135), (132))

$$\sqrt{-g}\,\vartheta_i^{\ k} = \frac{1}{2\varkappa}\{\Gamma^k_{lm}(\sqrt{-g}\,g^{lm})_{,\,i} - \Gamma^s_{ms}(\sqrt{-g}\,g^{km})_{,\,i} - \delta_i^k\,\mathfrak{L}\} \qquad (C.18)$$

$$\mathfrak{L} = \sqrt{-g}\,g^{ik}(\Gamma^r_{ik}\Gamma^l_{rl} - \Gamma^r_{il}\Gamma^l_{kr}). \qquad (C.19)$$

From (4.1-2) and (C.10) we get to the first order

$$\left.\begin{aligned} \Gamma^i_{kl} &= \frac{g^{ir}}{2}(g_{rk,\,l} + g_{rl,\,k} - g_{kl,\,r}) \\ &= \frac{\eta^{ir}}{2}(y_{rk,\,l} + y_{rl,\,k} - y_{kl,\,r}) \\ &= \frac{1}{2}\;(y^i_{k,\,l} + y^i_{l,\,k} - y_{kl}{}^{,\,i}), \end{aligned}\right\} \qquad (C.20)$$

$$\Gamma_{k\,i}^{\,i} = \frac{1}{2}(y^i_{k,\,i} + y_{,\,k} - y_{ki},{}^{i}) = \frac{1}{2}\,y_{,\,k}, \tag{C.21}$$

$$(\sqrt{-g}\,g^{lm})_{,\,i} = \left[\left(1+\frac{1}{2}\,y\right)(\eta^{lm} - y^{lm})\right]_{,\,i} = \left(\frac{1}{2}\,\eta^{lm}\,y - y^{lm}\right)_{,\,i}. \tag{C.22}$$

Thus we get to the second order in harmonic coordinates, using (C.18-22) and (4.24),

$$\begin{aligned}\vartheta_i^{\,k} &= \frac{1}{2\,\varkappa}\left\{\frac{1}{2}\,(y^k_{l,\,m} + y^k_{m,\,l} - y_{lm},{}^{k})\left(\frac{1}{2}\,\eta^{lm}\,y - y^{lm}\right)_{,\,i} - \frac{1}{2}\,y_{,\,m}\left(\frac{1}{2}\,\eta^{km}\,y - y^{km}\right)_{,\,i} - \delta^k_i\,\mathfrak{L}\right\}\\ &= \frac{1}{4\,\varkappa}\left\{\left(y^{kl}{}_{,\,l} - \frac{1}{2}\,y,{}^{k}\right)y_{,\,i} - (2\,y^k_{l,\,m} - y_{lm},{}^{k})\,y^{lm}{}_{,\,i} - \frac{1}{2}\,y,{}^{k}\,y_{,\,i} + y_{,\,m}\,y^{km}{}_{,\,i} - 2\,\delta^k_i\,\mathfrak{L}\right\},\end{aligned}$$

i. e.

$$\vartheta_i^{\,k} = \frac{1}{4\,\varkappa}\left\{(y_{lm},{}^{k} - 2\,y^k_{l,\,m})\,y^{lm}{}_{,\,i} - \frac{1}{2}\,y,{}^{k}\,y_{,\,i} + y_{,\,m}\,y^{km}{}_{,\,i} - 2\,\delta^k_i\,\mathfrak{L}\right\}. \tag{C.23}$$

Further,

$$\begin{aligned}\mathfrak{L} &= \frac{\eta^{ik}}{4}\,[(y^r_{i,\,k} + y^r_{k,\,i} - y_{ik},{}^{r})\,y_{,\,r} - (y^r_{i,\,l} + y^r_{l,\,i} - y_{il},{}^{r})(y^l_{k,\,r} + y^l_{r,\,k} - y_{kr},{}^{l})]\\ &= \frac{1}{4}\,[(2\,y^{rs}{}_{,\,s} - y,{}^{r})\,y_{,\,r} - (y^r_{s,\,t} + y^r_{t,\,s} - y_{st},{}^{r})(y^{st}{}_{,\,r} + y^{t,\,s}_r - y^{s,\,t}_r)].\end{aligned}$$

Here, the first term is zero on account of (4.24) and, since the first factor in the last term is symmetrical in s and t, we finally get

$$\mathfrak{L} = \frac{1}{4}\,y_{rs,\,t}\,y^{rs,\,t} - \frac{1}{2}\,y_{rs,\,t}\,y^{ts,\,r}. \tag{C.24}$$

A comparison of (C.23-24) with (4.53-54) shows that $\vartheta_i^{\,k}$ and $t_i^{\,k}$ in general are different already in the terms of second order. For the difference we get by (C.23) and (4.53)

$$\vartheta_i^{\,k} - t_i^{\,k} = \frac{1}{4\,\varkappa}\left\{-\,y^k_{l,\,m}\,y^{lm}{}_{,\,i} + \frac{1}{2}\,y_{,\,l}\,y^{kl}{}_{,\,i} - 2\,\delta^k_i\,(\mathfrak{L} - \hat{\mathfrak{L}})\right\} \tag{C.25}$$

and by (C.24) and (4.54)

$$\mathfrak{L} - \hat{\mathfrak{L}} = -\frac{1}{4}\left[y_{rs,\,t}\,y^{ts,\,r} - \frac{1}{4}\,y_{,\,r}\,y,{}^{r}\right] = -\frac{1}{4}\,[(y^{mn}\,y^l_n)_{,\,m} - y^{lm}\,y_{,\,m})_{,\,l}, \tag{C.26}$$

as is seen at once by performing the differentiations in the last expression (C.26) and using (4.24). Thus, $\mathfrak{L} - \hat{\mathfrak{L}}$ has the form of a usual divergence and, as we shall see, this is also the case for the two first terms in (C.25). In fact, they may be written

$$-\,y^k_{m,\,l}\,y^{lm}{}_{,\,i} + \frac{1}{2}\,y_{,\,m}\,y^{km}{}_{,\,i} = \left[-\,y^k_m\,y^{lm}{}_{,\,i} + \frac{1}{2}\,y^{km}\,y_{,\,m}\,\delta^l_i\right]_{,\,l},$$

where again use has been made of (4.24). Hence

$$\vartheta_i^{\,k} - t_i^{\,k} = \frac{1}{4\varkappa}\left[-y_m^k\, y^{lm}{}_{,\,i} + \frac{1}{2}\,\delta_i^k\,(y^{mn}\,y_n^l)_{,\,m} + \frac{1}{2}\,\delta_i^l\, y^{km}\, y_{,\,m} - \frac{1}{2}\,\delta_i^k\, y^{lm}\, y_{,\,m}\right]_{,\,l}. \tag{C.27}$$

The last two terms inside the brackets give a contribution which is antisymmetrical in k and l. We shall see now that also the two first terms can be brought into the form of a superpotential, for we have

$$\left[-y_m^k\, y^{lm}{}_{,\,i} + \frac{1}{2}\,\delta_i^{\,k}\,(y^{mn}\,y_n^l)_{,\,m}\right]_{,\,l} = \left[-(y_m^k\, y^{lm})_{,\,i} + y_{m,\,i}^k\, y^{lm} + \frac{1}{2}\,\delta_i^{\,k}\,(y^{mn}\,y_n^l)_{,\,m}\right]_{,\,l}$$

$$= \frac{1}{2}\left\{(y_{m,\,i}^k\, y^{lm} - y_m^k\, y^{lm}{}_{,\,i})_{,\,l} + \delta_i^{\,k}\,(y^{mn}\,y_n^l)_{,\,m,\,l} - (y_n^k\, y^{ln})_{,\,m,\,l}\,\delta_i^m\right\}$$

$$= \frac{1}{2}\left[y_{m,\,i}^k\, y^{lm} - y^{km}\, y_{m,\,i}^l + \delta_i^{\,k}\,(y^{mn}\,y_n^l)_{,\,m} - \delta_i^l\,(y_n^k\, y^{mn})_{,\,m}\right]_{,\,l}.$$

Thus, we see that the equation (C.27) for the difference $\vartheta_i^{\,k} - t_i^{\,k}$ can be brought into the form

$$\vartheta_i^{\,k} - t_i^{\,k} = X_i^{\,kl}{}_{,\,l} \tag{C.28}$$

with the superpotential

$$\left.\begin{aligned} X_i^{\,kl} = -X_i^{\,lk} = \frac{1}{8\varkappa}\Big\{ & y_{m,\,i}^k\, y^{lm} - y_{m,\,i}^l\, y^{km} + \delta_i^{\,k}\,[(y^{mn}\,y_n^l)_{,\,m} - y^{lm}\, y_{,\,m}] \\ & - \delta_i^l\,[(y^{mn}\,y_n^k)_{,\,m} - y^{km}\, y_{,\,m}]\Big\}. \end{aligned}\right\} \tag{C.29}$$

We knew beforehand that this should be possible, since we have exactly

$$\sqrt{-g}\,(\vartheta_i^{\,k} - t_i^{\,k}) = \Theta_i^{\,k} - \mathrm{T}_i^{\,k} = (h_i^{\,kl} - \mathfrak{U}_i^{\,kl})_{,\,l}.$$

However, it is interesting that the second order part $X_i^{\,kl}$ of the superpotential $h_i^{\,kl} - \mathfrak{U}_i^{\,kl}$ depends (quadratically) on the first order functions y_{ik}, only.

Appendix D

The covariant derivative of the second kind of a tensor $A_{ik}{}^l$ of rank 3 is, by generalization of (5.10),

$$A_{ik}{}^l{}_{|\,m} = A_{ik}{}^l{}_{;\,m} - A_{rk}{}^l\,\gamma^r{}_{im} - A_{ir}{}^l\,\gamma^r{}_{km} + A_{ik}{}^r\,\gamma^l{}_{rm}. \tag{D.1}$$

If we apply this rule to the tensor $\gamma_{ik}{}^l$ and make a contraction of the indices l and m, we get

$$\gamma_{ik}{}^l{}_{|\,l} = \gamma_{ik}{}^l{}_{;\,l} - \gamma_{rk}{}^l\,\gamma^r{}_{il} - \gamma_{ir}{}^l\,\gamma^r{}_{kl} + \gamma_{ik}{}^r\,\gamma^l{}_{rl}. \tag{D.2}$$

On account of the symmetry relation $\gamma_{ir}{}^{l} = -\gamma_{ri}{}^{l}$ the second and the third term on the right-hand side of (D.2) cancel. Therefore, we get by (3.10) and (3.18)

$$\gamma_{ik}{}^{l}{}_{|l} = \gamma_{ik}{}^{l}{}_{;l} + \gamma_{ikl}\Phi^{l} = \xi_{ik}, \tag{D.3}$$

i. e. the equation (5.11) in the text.

From (5.7) and (5.10) we get

$$\left.\begin{aligned} \underset{a}{h}_{k;l} &= \underset{a}{h}_{k|l} + \underset{a}{h}_{r}\gamma^{r}{}_{kl} = \underset{a}{h}_{r}\gamma^{r}{}_{kl} \\ \underset{a}{h}_{k;l|m} &= \underset{a}{h}_{r}\gamma^{r}{}_{kl|m} \end{aligned}\right\} \tag{D.4}$$

and

$$\underset{a}{h}_{k;l;m} = \underset{a}{h}_{k;l|m} + \underset{a}{h}_{s;l}\gamma^{s}{}_{km} + \underset{a}{h}_{k;s}\gamma^{s}_{lm}$$

which by (D.4) may be written

$$\underset{a}{h}_{k;l;m} = \underset{a}{h}_{r}\left[\gamma^{r}{}_{kl|m} + \gamma^{r}{}_{sl}\gamma^{s}{}_{km} + \gamma^{r}{}_{ks}\gamma^{s}{}_{lm}\right]. \tag{D.5}$$

By means of (A.2) we then get for the Riemann curvature tensor

$$R^{i}{}_{klm} = \gamma^{i}{}_{km|l} - \gamma^{i}{}_{kl|m} + \gamma^{i}{}_{sm}\gamma^{s}{}_{kl} - \gamma^{i}{}_{sl}\gamma^{s}{}_{km} + \gamma^{i}{}_{ks}(\gamma^{s}{}_{ml} - \gamma^{s}{}_{lm}). \tag{D.6}$$

Contraction with respect to the indices i and l gives the contracted curvature tensor

$$R_{km} = \gamma^{r}{}_{km|r} - \gamma^{r}{}_{kr|m} + \gamma^{r}{}_{sm}\gamma^{s}{}_{kr} - \gamma^{r}{}_{sr}\gamma^{s}{}_{km} + \gamma^{r}{}_{ks}(\gamma^{s}{}_{mr} - \gamma^{s}{}_{rm})$$

or, by (3.10),

$$R_{km} = \gamma^{r}{}_{km|r} - \Phi_{k|m} - \Phi_{s}\gamma^{s}{}_{km} + \gamma^{r}{}_{ks}\gamma^{s}{}_{mr}. \tag{D.7}$$

By further contraction we get the curvature scalar

$$R = R_{k}{}^{k} = -2\Phi^{r}{}_{|r} + \Phi_{r}\Phi^{r} + \gamma_{rst}\gamma^{tsr}. \tag{D.8}$$

The equations (D.7-8) are identical with the equations (5.12) in the text.

If $\psi(x)$ is a scalar field, we have by definition

$$\psi_{|i} = \psi_{;i} = \psi_{,i}. \tag{D.9}$$

Then, if A_i is a vector field, we can form four scalar fields $\overset{a}{A} = \overset{a}{h}{}^{i} A_i$ (the local tetrad components) from which we get four vector fields by the operation (D.9), i. e.

$$\overset{a}{A}_{,k} = \overset{a}{A}_{|k} = \overset{a}{h}{}^{i}{}_{|k} A_i + \overset{a}{h}{}^{i} A_{i|k} = \overset{a}{h}{}^{i} A_{i|k}. \tag{D.10}$$

Here we have used the equations (5.7) and the rule for covariant differentiation of products of tensors. From the vector fields (D.10) we can form tensors by further covariant differentiations:

$$\overset{a}{A}_{|k|l} = \overset{a}{A}_{|k,l} - \overset{a}{A}_{|r}\Delta^r{}_{kl} = \overset{a}{A}_{,k,l} - \overset{a}{h}{}^i A_{i|r}\Delta^r{}_{kl}, \tag{D.11}$$

where we have used (D.10). On the other hand, we get by differentiation of (D.10)

$$\overset{a}{A}_{|k|l} = \overset{a}{h}{}^i{}_{|l} A_{i|k} + \overset{a}{h}{}^i A_{i|k|l} = \overset{a}{h}{}^i A_{i|k|l}. \tag{D.12}$$

Hence,

$$\overset{a}{h}{}^i (A_{i|k|l} + A_{i|r}\Delta^r{}_{kl}) = \overset{a}{A}_{,k,l}$$

or

$$A_{i|k|l} + A_{ir}\Delta^r{}_{kl} = \underset{a}{h}_i \overset{a}{A}_{,k,l}. \tag{D.13}$$

If we subtract from (D.13) the equation obtained by interchanging k and l we get, since $\overset{a}{A}_{,k,l} = \overset{a}{A}_{,l,k}$,

$$A_{i|k|l} - A_{i|l|k} = -A_{i|r}\Lambda^r{}_{kl}, \tag{D.14}$$

where we have introduced the tensor

$$\left.\begin{aligned} \Lambda^i{}_{kl} &\equiv \Delta^i{}_{kl} - \Delta^i{}_{lk} = \overset{a}{h}{}^i \left(\underset{a}{h}_{k,l} - \underset{a}{h}_{l,k}\right) \\ &= \overset{a}{h}{}^i \left(\underset{a}{h}_{k;l} - \underset{a}{h}_{l;k}\right) = \gamma^i{}_{kl} - \gamma^i{}_{lk} = -\Lambda^i{}_{lk}. \end{aligned}\right\} \tag{D.15}$$

The equation (D.14) is the commutation law for covariant differentiations of the second kind of a vector field. It is seen at once that a similar equation holds for any tensor $A_{i_1 i_2 \cdots i_n}$ with an arbitrary number of indices, i. e. we have

$$A_{i_1 i_2 \cdots i_n|k|l} - A_{i_1 i_2 \cdots i_n|l|k} = -A_{i_1 i_2 \cdots i_n|r}\Lambda^r{}_{kl}. \tag{D.16}$$

In order to prove (D.16) we have simply to form the scalar fields

$$\overset{a_1 a_2 \cdots a_n}{A} = \overset{a_1}{h}{}^{i_1} \overset{a_2}{h}{}^{i_2} \cdots\cdots \overset{a_n}{h}{}^{i_n} A_{i_1 i_2 \cdots i_n}$$

and repeat the operations performed in (D.10-14) on these scalars. We also get from (5.5), for any vector field,

$$A_{i|k} - A_{k|i} = A_{i,k} - A_{k,i} - A_r\Lambda^r{}_{ik}. \tag{D.17}$$

Covariant differentiation of (D.14) gives

$$A_{i|k|l|m} - A_{i|l|k|m} = -A_{i|r|m}\Lambda^r{}_{kl} - A_{i|r}\Lambda^r{}_{kl|m}. \tag{D.18}$$

Adding to this equation the two equations obtained by cyclic permutation of the indices k, l, m, we get

$$\begin{aligned} &(A_{i|k|l|m} - A_{i|k|m|l}) + (A_{i|l|m|k} - A_{i|l|k|m}) + (A_{i|m|k|l} - A_{i|m|l|k}) \\ &= -A_{i|r|m}\Lambda^r{}_{kl} - A_{i|r|k}\Lambda^r{}_{lm} - A_{i|r|l}\Lambda^r{}_{mk} - A_{i|r}(\Lambda^r{}_{kl|m} + \Lambda^r{}_{lm|k} + \Lambda^r{}_{mk|l}) \end{aligned}$$

which, by means of (D.16), can be written

$$(A_{i\,|\,r\,|\,k}-A_{i\,|\,k\,|\,r})\Lambda^r{}_{lm}+(A_{i\,|\,r\,|\,l}-A_{i\,|\,l\,|\,r})\Lambda^r{}_{mk}+(A_{i\,|\,r\,|\,m}-A_{i\,|\,m\,|\,r})\Lambda^r{}_{kl}$$
$$=-A_{i\,|\,r}(\Lambda^r{}_{kl\,|\,m}+\Lambda^r{}_{lm\,|\,k}+\Lambda^r{}_{mk\,|\,l}).$$

By application of (D.14) this equation takes the form

$$A_{i\,|\,r}\,[\Lambda^r{}_{kl\,|\,m}+\Lambda^r{}_{lm\,|\,k}+\Lambda^r{}_{mk\,|\,l}-\Lambda^r{}_{sk}\Lambda^s{}_{lm}-\Lambda^r{}_{sl}\Lambda^s{}_{mk}-\Lambda^r{}_{sm}\Lambda^s{}_{kl}]=0. \qquad \text{(D.19)}$$

Since this equation must hold for any vector field A_i, the bracket in (D.19) must be identically zero. Thus, we get the identity

$$\Lambda^i{}_{kl\,|\,m}+\Lambda^i{}_{lm\,|\,k}+\Lambda^i{}_{mk\,|\,l}+\Lambda^i{}_{kr}\Lambda^r{}_{lm}+\Lambda^i{}_{lr}\Lambda^r{}_{mk}+\Lambda^i{}_{mr}\Lambda^r{}_{kl}=0. \qquad \text{(D.20)}$$

From (D.15), (3.10) we get

$$\Lambda^i{}_{ki}=-\Lambda^i{}_{ik}=\gamma^i{}_{ki}=\Phi_k. \qquad \text{(D.21)}$$

Thus, by contraction of (D.20) we obtain the identity

$$\Lambda^r{}_{kl\,|\,r}+\Phi_{l\,|\,k}-\Phi_{k\,|\,l}-\Phi_r\Lambda^r{}_{kl}=0 \qquad \text{(D.22)}$$

or

$$\Lambda^r{}_{kl\,|\,r}-\Phi_{k,\,l}+\Phi_{l,\,k}=0 \qquad \text{(D.23)}$$

by (D.17) applied to the vector field Φ_k. (D.23) is the identity (5.13) which shows that the contracted curvature tensor R_{ik} given by (D.7) or (5.12) is symmetrical. Similarly, it is easily seen that the well-known relation

$$R^i{}_{klm}+R^i{}_{lmk}+R^i{}_{mkl}=0, \qquad \text{(D.24)}$$

holding for the Riemann curvature tensor (D.6), is a consequence of the identity (D.20). The symmetry relations

$$R_{iklm}=-R_{kilm}=-R_{ikml}=R_{kiml} \qquad \text{(D.25)}$$

follow directly from (D.6) and the symmetry properties of γ_{ikl}.

Finally we note that the derivative of the determinant

$$h=\det\left\{\underset{a}{h}\right\},$$

by well-known rules for differentiation of determinants, is

$$h_{,\,i}=h\,\overset{a}{h}{}^k\,\underset{a}{h}{}_{k,\,i}=h\,\Delta^k{}_{ki}. \qquad \text{(D.26)}$$

Thus, the covariant divergence of the second kind of a vector field A^i is, by (5.5) and (D.26), (D.15), (D.21),

$$\left.\begin{aligned} A^i{}_{|i} &= A^i{}_{,i} + A^r \Delta^i{}_{ri} = A^i{}_{,i} + A^r (\Delta^i{}_{ir} + \Delta^i{}_{ri}) \\ &= A^i{}_{,i} + A^r \frac{h_{,r}}{h} + A^r \Phi_r, \end{aligned}\right\} \quad \text{(D.27)}$$

i. e.

$$A^i{}_{|i} = \frac{1}{h}(hA^i)_{,i} + A^i \Phi_i .$$

Similarly, for tensor fields of rank 2 and 3,

$$A_i{}^k{}_{|k} = \frac{1}{h}(hA_i{}^k)_{,k} + A_i{}^k \Phi_k - A_r{}^k \Delta^r{}_{ik}, \tag{D.28}$$

$$A_i{}^{kl}{}_{|l} = \frac{1}{h}(hA_i{}^{kl})_{,l} + A_i{}^{kl} \Phi_l - A_r{}^{kl} \Delta^r{}_{il} + A_i{}^{rl} \Delta^k{}_{rl}. \tag{D.29}$$

Thus, for the corresponding tensor density,

$$\mathfrak{A}_i{}^{kl}{}_{|l} \equiv |h| A_i{}^{kl}{}_{|l} = \mathfrak{A}_i{}^{kl}{}_{,l} + \mathfrak{A}_i{}^{kl} \Phi_l - \mathfrak{A}_r{}^{kl} \Delta^r{}_{il} + \mathfrak{A}_i{}^{rl} \Delta^k{}_{rl}. \tag{D.30}$$

In particular we get by (D.15) for the superpotential $\mathfrak{U}_i{}^{kl}$ which is antisymmetric in k and l

$$\mathfrak{U}_i{}^{kl}{}_{|l} = \mathfrak{U}_i{}^{kl}{}_{,l} + \mathfrak{U}_i{}^{kl} \Phi_l - \mathfrak{U}_r{}^{kl} \Delta^r{}_{il} + \frac{1}{2} \mathfrak{U}_i{}^{rl} \Lambda^k{}_{rl}. \tag{D.31}$$

Therefore the energy momentum complex $\mathsf{T}_i{}^k$ may also be written

$$\mathsf{T}_i{}^k = \mathfrak{U}_i{}^{kl}{}_{,l} = \mathfrak{U}_i{}^{kl}{}_{|l} - \mathfrak{U}_i{}^{kl} \Phi_l - \mathfrak{U}_i{}^{lm} \gamma^k{}_{lm} + \mathfrak{U}_m{}^{kl} \Delta^m{}_{il}. \tag{D.32}$$

Now, by (3.13) we have

$$\begin{aligned} \mathfrak{U}_i{}^{kl}{}_{|l} &= \frac{|h|}{\varkappa} [\gamma^{kl}{}_{i|l} - \delta_i^k \Phi^l{}_{|l} + \Phi^k{}_{|i}] \\ \mathfrak{U}_i{}^{kl} \Phi_l &= \frac{|h|}{\varkappa} [\gamma^{kl}{}_i \Phi_l - \delta_i^k \Phi^l \Phi_l + \Phi^k \Phi_i] \\ \mathfrak{U}_i{}^{lm} \gamma^k{}_{lm} &= \frac{|h|}{\varkappa} [\gamma^{lm}{}_i \gamma^k{}_{lm} - \gamma^k{}_{im} \Phi^m + \gamma^k{}_{li} \Phi^l] \\ &= \frac{|h|}{\varkappa} [\gamma_{lmi} \gamma^{klm} + \Lambda^k{}_{li} \Phi^l]. \end{aligned}$$

Thus,

$$\left.\begin{aligned} \mathsf{T}_i{}^k = \frac{|h|}{\varkappa} \{ & -\gamma^{lk}{}_{i|l} + \Phi^k{}_{|i} + \gamma^{lk}{}_i \Phi_l + \delta_i^k (-\Phi^l{}_{|l} + \Phi^l \Phi_l) \\ & - \Phi^k \Phi_i - \gamma_{lmi} \gamma^{klm} - \Lambda^k{}_{li} \Phi^l \} + \mathfrak{U}_m{}^{kl} \Lambda^m{}_{il}. \end{aligned}\right\} \quad \text{(D.33)}$$

Using (5.12) and the fact that $R_{ik} = R_{ki}$, we get after a simple calculation

$$\mathsf{T}_i^k = \frac{|h|}{\varkappa}\left\{ - R_i^k + \frac{1}{2}\delta_i^k R + \Lambda_{mli}\gamma^{klm} - \Phi_i \Phi^k + \Lambda^k{}_{il}\Phi^l - \frac{1}{2}\delta_i^k \hat{\mathfrak{L}}/|h|\right\} + \mathfrak{U}_m{}^{kl}\Delta^m{}_{il}$$

with $\hat{\mathfrak{L}}$ given by (3.12).
Hence,

$$\mathsf{T}_i^k = |h|\,[T_i^k + t_i^k]$$

with

$$t_i^k = \frac{1}{\varkappa}\left\{\Lambda_{mli}\gamma^{klm} - \Phi_i\Phi^k + \Lambda^k{}_{il}\Phi^l - \frac{1}{2\,|h|}\,\delta_i^k\,\hat{\mathfrak{L}}\right\} + U_m{}^{kl}\Delta^m{}_{il}. \qquad \text{(D.34)}$$

On account of (D.15), this expression is easily seen to be in accordance with the earlier expression (3.14). The first term in (D.34) is a tensor and, from the transformation law of the quantity $\Delta^i{}_{kl}$, we get at once the general transformation law for t_i^k. Under arbitrary transformations we have, in analogy with the Christoffel formula,

$$\left.\begin{aligned} \Delta'^m{}_{il} &= \overset{a}{h}{}'^m \underset{a}{h}'_{i,l} = \frac{\partial x'^m}{\partial x^r}\overset{a}{h}{}^r\left[\frac{\partial}{\partial x'^l}\left(\frac{\partial x^s}{\partial x'^i}\right)\cdot \underset{a}{h}_s + \frac{\partial x^s}{\partial x'^i}\frac{\partial x^t}{\partial x'^l}\underset{a}{h}_{s,t}\right] \\ &= \frac{\partial x'^m}{\partial x^r}\frac{\partial^2 x^r}{\partial x'^l\,\partial x'^i} + \frac{\partial x'^m}{\partial x^r}\frac{\partial x^s}{\partial x'^i}\frac{\partial x^t}{\partial x'^l}\Delta^r{}_{st}. \end{aligned}\right\} \qquad \text{(D.35)}$$

The last term corresponds to a tensor transformation and, since also $U_m{}^{kl}$ is a tensor, we get by (D.34-35)

$$t'^k_i = \frac{\partial x^l}{\partial x'^i}\frac{\partial x'^k}{\partial x^m}t_l^m + U'^{kl}_m\frac{\partial x'^m}{\partial x^r}\frac{\partial^2 x^r}{\partial x'^l\,\partial x'^i} \qquad \text{(D.36)}$$

or

$$t'^k_i = \frac{\partial x'^k}{\partial x^m}\left[\frac{\partial x^l}{\partial x'^i}t_l^m + \frac{\partial}{\partial x^n}\left(\frac{\partial x^r}{\partial x'^i}\right)\cdot U_r{}^{mn}\right] \qquad \text{(D.37)}$$

in accordance with an unpublished result by C. Pellegrini. From (D.37) it follows that t_i^i is a scalar, and in fact one gets from (D.34)

$$t_i^i = \frac{1}{\varkappa}(\Phi_r\Phi^r - \gamma_{rst}\gamma^{tsr}) = -\,\hat{\mathfrak{L}}/\varkappa\,|h|. \qquad \text{(D.38)}$$

Also one sees that t_4^k transforms as a vector under the group of purely spatial transformations (2.3).

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